Gödel Logics – a short survey

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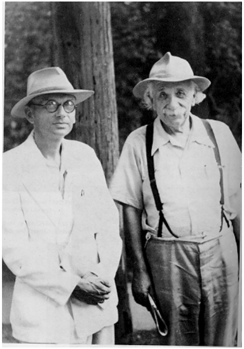
Today's program

- Development of many-valued logics
  t-norm based logics
- Gödel logics (propositional
  quantified propositional, first order)
- Gödel logics and ...
  - Topology
  - Order Theory
  - Computability
- Other topics
  - Kripke frames and beyond the reals
  - Monadic fragment
  - Proof theory
- History
- Conclusion

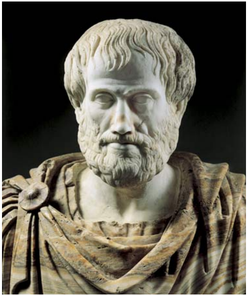


Gödel with a somehow famous physicist

Development of many-valued logics

The most important stops

- Platon, Aristoteles (De Interpretatione IX), Ockham: *future possibilities*, problem of determination vs. fatalism.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- Post 1920: Many-valued logic dealing with functional completion

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- Gödel 1932: Finite valued logics for approximation of intuitionistic logic
- Bočvar 1938: Logic of *Paradoxa*
- Kleene 1952: Logic of the *unknown*
- Zadeh 1965: Fuzzy sets and fuzzy logics

How do we continue?

Arbitrary finite-valued logics

For all finite-valued logics with truth-value functions there is an automatic algorithm for generating a sequent calculus, proving completeness etc (MultLog, MultSeq: Baaz, Fermüller, Salzer, Zach et al. 1996ff).

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Infinite valued logics

Does it make sense to take truth values from arbitrary partial orderings?

⇒ No, because every logics with substitution property would be a many-valued logic!

Take all sentences as truth values, and all sentences of the logic as designated truth values.

Design decisions

Basic requirements

- Extension of classical logic
- $[0, 1]$ as super-set of the truth value set
- functional relation between the truth value of a formula and the one of its sub-formulas.

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Additional 'natural' properties of the conjunction

- associative $((A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C))$
- commutative $(A \wedge B \Leftrightarrow B \wedge A)$
- order preserving
  If A is less true than b, then $A \wedge C$ is less (or equal) true than $B \wedge C$.
- continuous

Definition of (continuous) t-norms

Definition

A t-norm is an associative, commutative, and monotone mapping from $[0,1]^2 \to [0,1]$ with 1 as neutral element.

- $(x \star y) \star z = x \star (y \star z)$
- $x \star y = y \star x$
- $x \leqslant y \supset x \star z \leqslant y \star z$
- $1 \star x = x$
- $\star$ is continuous

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Algebraic view

$\langle [0,1], \star, 1, \leqslant \rangle$ is a commutative and ordered monoid.

From t-norm to the logic

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The residuum of a t-norm

Every t-norm has a residuum

$$x \star z \leqslant y \quad \Leftrightarrow \quad z \leqslant (x \Rightarrow y)$$

$$x \Rightarrow y \quad := \quad \max\{z : x \star z \leqslant y\}$$

From t-norm to the logic

The residuum of a t-norm

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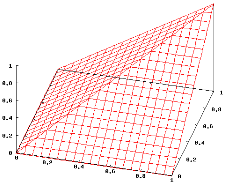
Truth functions for operators

- strong conjunction &: defined via the t-norm
- implication $\supset$: defined via the residuum
- Negation: $\neg A := A \supset \bot$
- (weak) disjunction: $A \vee B := (A \supset B) \supset B$
- (weak) conjunction: $A \wedge B := \neg(\neg A \vee \neg B)$
- strong disjunction: $A \bigvee B := \neg(A \supset \neg B)$

Basic t-norms

Gödel



non-trivial idempotent elements

Basic t-norms

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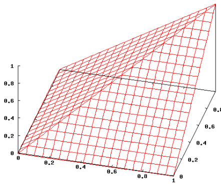

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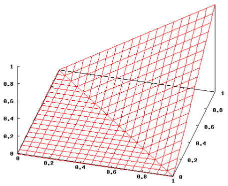

no non-trivial idempotent elements, but zero divisors

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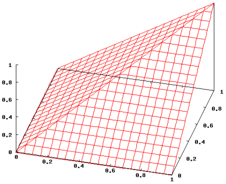

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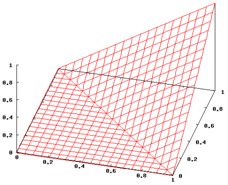

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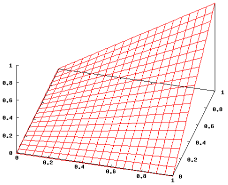
Product



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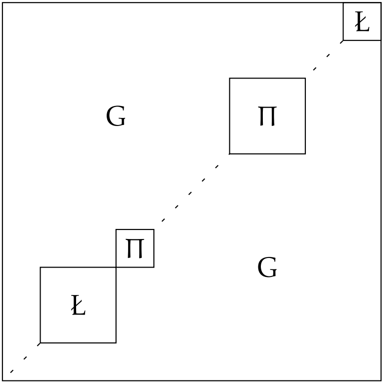
Representation of t-norm

Theorem (Mostert and Shields, 1957)

Every t*-norm is the ordinal sum of Łukasiewicz* t*-norm and Product* t*-norm.*



Questions and results

- Basic logic: the logic of all t-norms (Hajek 1998)
- Axiomatizability: propositional logic: easy, first-order: only Gödel logics are axiomatizable (Scarpellini 1962, Horn 1969, Takeuti, Titani 1984, Takano 1987)
- calculi for propositional logic: sequent calculus for Gödel logic (Avron 1991, *hyper sequent calculus*), Π- und Ł-Logik (Gabbay, Metcalfe, Olivetti 2003).
- calculi for first order logic: only for Gödel logic (Baaz, Zach 2000)
- Game interpretation: Łukasiewicz Logic: interpretation via Ulam's games (Mundici, 1991-93), Gödel, Product, Łukasiewicz: interpretation via Lorenzen style games (Giles 1970; Fermüller, Metcalfe, Ciabattoni 2003-04)
- other questions: automatic theorem proving, size of families, …

Gödel Logics

Propositional Logics

Propositional logic

Usual propositional language, $\neg A$ is defined as $A \supset \bot$.

Evaluations

Fix a truth value set $\{0, 1\} \subseteq V \subseteq [0, 1]$
v maps propositional variables to elements of V

$$v(A \wedge B) = \min\{v(A), v(B)\}$$
$$v(A \vee B) = \max\{v(A), v(B)\}$$
$$v(A \supset B) = \begin{cases} v(B) & \text{if } v(A) > v(B) \\ 1 & \text{if } v(A) \leqslant v(B). \end{cases}$$

Negation

This yields the following definition of the semantics of $\neg$:

$$v(\neg A) = \begin{cases} 0 & \text{if } v(A) > 0 \\ 1 & \text{otherwise} \end{cases}$$

Takeuti's observation

Gödel implication

$$v(A \supset B) = \begin{cases} v(B) & \text{if } v(A) > v(B) \\ 1 & \text{if } v(A) \leqslant v(B). \end{cases}$$

is the only one satisfying:

- $v(A) \leqslant v(B) \Leftrightarrow v(A \supset B) = 1$
- $\Pi \cup \{A\} \Vdash B \Leftrightarrow \Pi \Vdash A \supset B$
- $\Pi \Vdash B \Rightarrow \min\{v(A) : A \in \Pi\} \leqslant v(B)$
  (and if $\Pi = \emptyset \Rightarrow 1 \leqslant v(B)$)

Definition of the logic

$$G_V = \{A : \forall v \text{ into } V : v(A) = 1\}$$

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$$G_V = \{A : \forall v \text{ into } V : v(A) = 1\}$$

Examples

$$
\begin{array}{ll}
V = \{0, 1\} & \to G_V = \mathrm{CPL} \\
V_1 = \{0, 1/2, 1\}, V_2 = \{0, 1/3, 1\} & \to G_{V_1} = G_{V_2} \\
V_\uparrow = \{1 - 1/n : n \geqslant 1\} \cup \{1\} & \to G_{V_\uparrow} = G_\uparrow \\
V_\downarrow = \{1/n : n \geqslant 1\} \cup \{0\} & \to G_{V_\downarrow} = G_\downarrow
\end{array}
$$

Propositional completeness

- Lindenbaum algebra of the formulas
- show that the algebra $\mathcal{F}/\equiv$ is a subalgebra of

$$X = \prod_{i=1}^{n!} \mathcal{C}(\bot, \pi_i(p_1, \ldots, p_n), \top)$$

($\mathcal{C}(\ldots)$ being the chain consisting of the listed elements, and the π_i all the permutations) by defining

$$\phi(|\alpha|) = (|\alpha|_{\mathcal{C}_1}, \ldots, |\alpha|_{\mathcal{C}_{n!}})$$

Consequences

- countably many propositional Gödel logics

$$G_2 \supset G_3 \supset \ldots \supset G_n \supset \ldots \supset G_\uparrow = G_\downarrow = G_V = G_\infty = \bigcap_{n \geqslant 2} G_n$$

  (where V is any infinite truth value set)

- if $f : V_1 \mapsto V_2$ with $f(0) = 0$ and $f(1) = 1$, order-preserving ($x < y \Rightarrow f(x) < f(y)$), then

$$G_{V_1} \supseteq G_{V_2}$$

- check on satisfiability and validity

Quantified Propositional Logics

Propositional Logic

Fix a truth value set $\{0, 1\} \subseteq V \subseteq [0, 1]$
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Quantified Propositional Logic

Fix a truth value set $\{0, 1\} \subseteq V \subseteq [0, 1]$, V closed
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$$v(\forall p A(p)) = \inf\{v(A(p)) : p \in P\}$$

$$v(\exists p A(p)) = \sup\{v(A(p)) : p \in P\}$$

Properties

	$V_{\downarrow}$	$V_{\uparrow}$	V_{∞}
Decidability	S1S	S1S	QE
Axiomatisation	HS+$\circ$	HS+$\circ$	HS, GS
QE	with $\circ$	with $\circ$	yes

(Baaz, Veith, Zach, P. 2000–)

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- uncountably many different quantified propositional logics (coding the topological structure)
- $G^{qp}_\uparrow = \bigcap_{n\in\mathbb{N}} G^{qp}_n$
- $\bigcap_{V\subseteq[0,1]} G^{qp}_V$ is not a quantified propositional Gödel logic (in contrast to propositional and first-order Gödel logics)

First Order Logics

First Order Gödel Logics

Fix a truth value set $\{0, 1\} \subseteq V \subseteq [0, 1]$, V closed

Interpretation v consists of

- a nonempty set U, the universe of v
- for each k-ary predicate symbol P a function $P^v : U^k \to V$
- for each k-ary function symbol f, a function $f^v : U^k \to U$
- for each variable x an object $x^v \in U$

Semantic cont.

Extend the valuation to all formulas

$$v(A \wedge B) = \min\{v(A), v(B)\}$$

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Horn-Takeuti-Titani-Takano – axiomatizability

Axiomatizability of $G_{[0,1]}$:

$$\text{LIN:} \quad A \supset B \vee B \supset A$$

$$\text{QS:} \quad \forall x(A(x) \vee B) \supset (\forall x A(x) \vee B)$$

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- Horn (1969)
  logic with truth values in a linearly ordered Heyting algebra
- Takeuti-Titani (1984), Takano (1987)
  intuitionistic fuzzy logic

Takano's proof

- set of formulas $\mathcal{F}$, equivalence relation $\equiv$ by provable equivalence
- show that $\mathcal{F}/\equiv$ is a (linear) Gödel algebra
- embed $\mathcal{F}/\equiv$ into $[0, 1]$
- show that embedding preserves infima and suprema (order-theoretic infima versus topological infima)

Connections

Gödel Logics and …

- topology
- order theory
- computation

Gödel Logics and Topology

Possible truth value sets

Perfect set

A set $P \subseteq \mathbb{R}$ is perfect if it is closed and all its points are limit points in P.

Cantor-Bendixon

Any closed $V \subseteq \mathbb{R}$ can be uniquely written as $V = P \cup C$, with P a perfect subset of V and C countable and open.

Possible truth value sets

Perfect set

A set $P \subseteq \mathbb{R}$ is perfect if it is closed and all its points are limit points in P.

Cantor-Bendixon

Any closed $V \subseteq \mathbb{R}$ can be uniquely written as $V = P \cup C$, with P a perfect subset of V and C countable and open.

Examples for perfect sets

- $[0, 1]$, any closed interval, any finite union of closed intervals
- Cantor Middle Third set $\mathbb{C}$: all numbers of $[0, 1]$ that do not have a 1 in the triadic notation (cut out all open middle intervals recursively) (perfect but nowhere dense)

The $\triangle$ operator

$$\triangle(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

- introduced and axiomatised by Takeuti and Titani in their discussion of intuitionistic fuzzy logic
- Baaz introduced and axiomatised in the context of Gödel logics
- parallels the 'recognizability' of 0, i.e., makes 1 recognizable.
- axiomatization of Gödel logics with $\triangle$ using Hilbert style calculus

Axiomatisation of $\triangle$

Baaz gave the following Hilbert style axiomatisation of the $\triangle$ operator:

$$
\begin{array}{ll}
\triangle 1 & \triangle A \vee \neg \triangle A \\
\triangle 2 & \triangle (A \vee B) \supset (\triangle A \vee \triangle B) \\
\triangle 3 & \triangle A \supset A \\
\triangle 4 & \triangle A \supset \triangle \triangle A \\
\triangle 5 & \triangle (A \supset B) \supset (\triangle A \supset \triangle B) \\
\triangle R & A \vdash \triangle A
\end{array}
$$

Extension of the interpretation:

$$
v(\triangle A) = \begin{cases} 1 & \text{if } v(A) = 1 \\ 0 & \text{if } v(A) < 1 \end{cases}
$$

Full characterization of $\triangle$-Axiomatizability

Recursively axiomatizable

- finitely valued
- V contains a perfect subset P and for both 0 and 1 it holds that they are either in the perfect kernel or isolated (4 cases)

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Recursively axiomatizable

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Not recursively enumerable

- countably infinite truth value set
- either 0 or 1 is not isolated but not in the perfect kernel

Axiomatizability

	$VAL_V^{\triangle}$	$(1\text{-}SAT_V^{\triangle})^c$	$(0^*\text{-}SAT_V^{\triangle})^c$		
with $\triangle$	uncountable			countable inf	finite
	$1 \in V^{\infty}$	1 isolated	otherwise		
$0 \in V^{\infty}$	re	re	not re	/	/
0 isolated	re	re	not re	not re	re
otherwise	not re	not re	not re	not re	/

	VAL_V	$(1\text{-}SAT_V)^c = (0^*\text{-}SAT_V)^c$	
without $\triangle$	uncountable	countable inf	finite
$0 \in V^{\infty}$	re	/	/
0 isolated	re	only $(1\text{-}SAT_V)^c$, $(0^*\text{-}SAT_V)^c$ re	re
otherwise	not re	not re	/

(Baaz, P., Zach 2007; Baaz, P. 2016)

Gödel Logics and Order Theory

Dummett – number of different logics

Dummett (1959)

All propositional (Gödel) logics based on infinite truth value sets coincide.
Thus, in total there are $\aleph_0$ different propositional logics.

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First order?

Lower bounds: always $\aleph_0$ (finitely valued, quantifier alterations, Cantor-Bendixon rank)

Counting first order logics

Comparing logic

If there is an injective, continuous and order preserving embedding from V_1 into V_2 that preserves 0 and 1, then $G_{V_1} \supseteq G_{V_2}$.

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Fraïssé Conjecture (1948), Laver (1971)

A $(Q, \leqslant)$ with reflexive and transitive $\leqslant$ is a *quasi-ordering*.

The set of scattered linear orderings ordered by embeddability is a *well-quasi-ordering* (does not contain infinite anti-chains nor infinitely descending chains)

Examples for quasi-orderings

Example

The collection of all linear orderings together with embeddability form a quasi-ordering, but not a partial ordering.

$\boldsymbol{\eta}$ and $\boldsymbol{\eta} + 1$ are different order types, but each embeddable into the other.

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Example

The collection of all linear orderings together with embeddability form a quasi-ordering, but not a partial ordering.

η and $\eta + 1$ are different order types, but each embeddable into the other.

Example

The collection of all linear orderings contain infinite descending chains, e.g. the order types of dense suborderings of $\mathbb{R}$.

Transfer to Gödel logics

Generalized Fraïssé Conjecture

The class of countable closed subsets of the reals with respect to injective and continuous embeddability is a well-quasi-ordering.

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Final result

The number of first order Gödel logics is $\aleph_0$.

(Beckmann, Goldstern, P. 2008)

Gödel Logics and Computation

Motivation

- Arnon Avron: *Hypersequents, Logical Consequence and Intermediate Logics for Concurrency* Ann.Math.Art.Int. 4 (1991) 225-248

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- Arnon Avron: *Hypersequents, Logical Consequence and Intermediate Logics for Concurrency* Ann.Math.Art.Int. 4 (1991) 225-248
  - *The second, deeper objective of this paper is to contribute towards a better understanding of the notion of logical consequence in general, and especially its possible relations with parallel computations*
  - *We believe that these logics [...] could serve as bases for parallel* λ*-calculi.*
  - *The name "communication rule" hints, of course, at a certain intuitive interpretation that we have of it as corresponding to the idea of exchanging information between two multiprocesses: [...]*

Setting the stage

$$\text{IL} \Leftrightarrow \text{LJ} \Leftrightarrow \text{ND} \overset{\text{Curry Howard}}{\Longleftrightarrow} \lambda$$

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Every proof system hides a model of computation.

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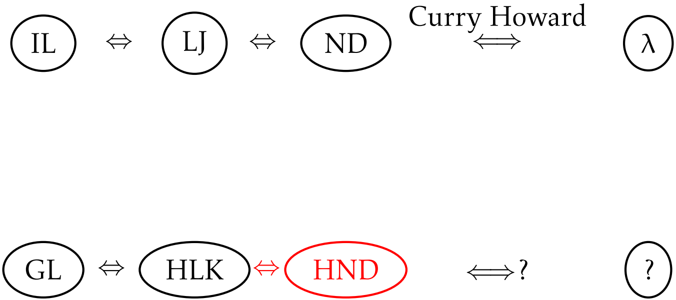

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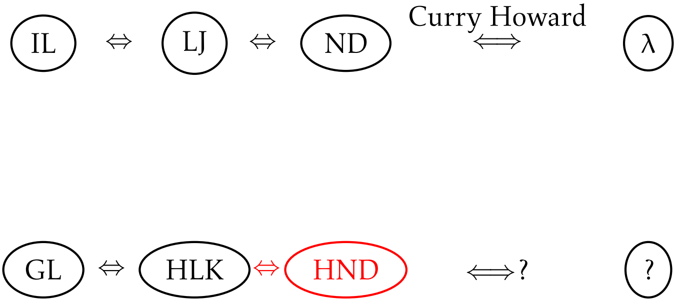

General aim: provide Curry-Howard style correspondences for parallel computation, starting from logical systems with good intuitive algebraic / relational semantics.

Wishlist

Properties we want to have:

(semi) local

- construction of deductions:
  apply ND inspired rules to extend a HND deductions
- modularity of deductions:
  reorder/restructure deductions
- analyticity (sub-formula property, ...)

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- construction of deductions:
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normalisation

- procedural normalisation via conversion steps

Our approach to Hyper Natural Deduction

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$$\text{(com)}\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}$$

$$\begin{array}{c}\Gamma \\ \vdots \\ A\end{array}$$

Our approach to Hyper Natural Deduction

$$(\text{com})\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}$$

$$\begin{array}{ccc} \Gamma & \qquad & \Delta \\ \vdots & & \vdots \\ A & & B \end{array}$$

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$$\begin{array}{ccc} \Gamma & & \Delta \\ \vdots & & \vdots \\ com\ \dfrac{A}{B} & & B \end{array}$$

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- consider sets of derivation trees
- divide communication (and split) into two dual parts
- search for minimal set of conditions that provides sound and complete deduction system

Reasoning in Hyper Natural Deduction

Double extension in *the spirit* of ND:

- from one tree to set of trees
- additional rules

Reasoning in Hyper Natural Deduction

Double extension in *the spirit* of ND:

- from one tree to set of trees
- additional rules

From $\begin{array}{c}\vdots\\ A\end{array}$ and $\begin{array}{c}\vdots\\ B\end{array}$ form $\mathrm{com}^{x}_{A,B}\ \dfrac{\begin{array}{c}\vdots\\ A\end{array}}{B}$ $\mathrm{com}^{\bar{x}}_{B,A}\ \dfrac{\begin{array}{c}\vdots\\ B\end{array}}{A}$

Reasoning in Hyper Natural Deduction

Double extension in *the spirit* of ND:

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$$\text{From} \quad \begin{array}{c}\vdots \\ A\end{array} \quad \text{and} \quad \begin{array}{c}\vdots \\ B\end{array} \quad \text{form} \quad \mathrm{com}^{x}_{A,B}\,\frac{\begin{array}{c}\vdots \\ A\end{array}}{B} \qquad \mathrm{com}^{\bar{x}}_{B,A}\,\frac{\begin{array}{c}\vdots \\ B\end{array}}{A}$$

$$\text{From} \quad \begin{array}{c}\Gamma, \Delta \\ \sigma \vdots \\ A\end{array} \quad \text{form} \quad x\colon \mathrm{Spt}_{\Gamma,\Delta}\,\frac{\begin{array}{c}[\Gamma], \Delta \\ \sigma \vdots \\ A\end{array}}{A} \qquad \bar{x}\colon \mathrm{Spt}_{\Delta,\Gamma}\,\frac{\begin{array}{c}\Gamma, [\Delta] \\ \sigma \vdots \\ A\end{array}}{A}$$

Results for Hyper Natural Deduction

- sound and complete for standard first order Gödel logic
- procedural normalization
- sub-formula property

(Beckmann, P. 2016)

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Beauty of this system

- Hyper rules – derivations are completely in ND style
- Hyper rules mimic HLK/BCF system
- natural style of deduction

Other topics

Gödel Logics and Kripke Frames

Gödel logic to Kripke frame

For each Gödel logic there is a countable linear Kripke frame such that the respective logics coincide.

Kripke frames to Gödel logic

For each countable linear Kripke frame there is a Gödel truth value set such that the respective logics coincide.

(Beckmann, P. 2007)

Going beyond $\mathbb{R}$

Takano (1987)

Axiomatization of the logic of linear Kripke frames based on $\mathbb{Q}$ (which is that of $G_{[0,1]}$).

Axiomatization of the logic of linear Kripke frames based on $\mathbb{R}$ needs an additional axiom.

Monadic Fragment

Decidability of validity and satisfiability

		validity	satisfiability
finite V	full monadic	Yes	Yes
infinite V with △	full monadic	No	No
	witnessed	No	No
	quantifer prefix	$\forall^*\exists^*$	(open!) $\exists^*\forall^*$
infinite V without △	full monadic	No	0 isolated: Yes 0 not isolated: No
	prenex	No	Yes
	$\exists$, $\neg$-free	No	Yes
	$\exists$	No	Yes
	$\neg$-free	No	Yes
	witnessed	No	Yes

(Baaz, Ciabattoni, P. 2011; Baaz, P. 2016)

Expressivity of Monadic logics

Take standard first order language.

Question: What can we express over complete linear orders?

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Question: What can we express over complete linear orders?

Same question with one (1) monadic predicate symbol?

The results

Theorem

If $0 \prec \alpha \prec \beta \prec \omega^\omega$ *with* $\beta \succeq \omega$, *then* $A_{\alpha,\beta} \in L(\alpha)$, *but* $A_{\alpha,\beta} \notin L(\beta)$.

The results

Theorem

If $0 \prec \alpha \prec \beta \prec \omega^\omega$ *with* $\beta \succeq \omega$, *then* $A_{\alpha,\beta} \in L(\alpha)$, *but* $A_{\alpha,\beta} \notin L(\beta)$.

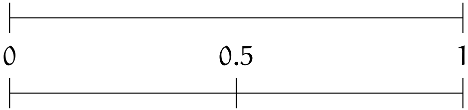
Theorem

If $0 \prec \alpha \prec \beta \prec \omega^\omega$, *then* $A^*_\alpha \in L(\alpha^*)$, *but* $A^*_\alpha \notin L(\beta^*)$.

(Beckmann, P 2014)

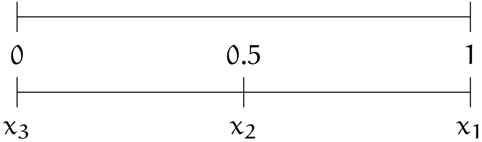
Basic idea

Separate 2 from 3-valued logic



Basic idea

Separate 2 from 3-valued logic



$$(x_1 \supset x_2) \vee (x_2 \supset x_3)$$

Proof theory

Hypersequent

Γ, Π finite multisets of formulas

$$\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$$

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Γ, Π finite multisets of formulas

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Rules

internal structural and logical (like LK)

external weakening and contraction

$$\frac{G \mid \Gamma, \Delta \Rightarrow A \quad G' \mid \Gamma', \Delta' \Rightarrow A'}{G \mid G' \mid \Gamma, \Delta' \Rightarrow A \mid \Delta, \Gamma' \Rightarrow A'} \ (\mathrm{com})$$

Calculus HG

Sound and completeness

HG is sound and complete for Gödel logics (propositional and first order)

(Avron 1992, Baaz, Zach 2000)

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$$\frac{\overline{\Rightarrow (A \supset B) \vee (B \supset A) \mid \Rightarrow (A \supset B) \vee (B \supset A)}}{\Rightarrow (A \supset B) \vee (B \supset A)}$$

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$$\dfrac{\dfrac{}{\Rightarrow A \supset B \mid \Rightarrow B \supset A}}{\dfrac{\Rightarrow (A \supset B) \vee (B \supset A) \mid \Rightarrow (A \supset B) \vee (B \supset A)}{\Rightarrow (A \supset B) \vee (B \supset A)}}$$

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$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow B \mid B \Rightarrow A}}{\Rightarrow A \supset B \mid \Rightarrow B \supset A}}{\dfrac{\Rightarrow (A \supset B) \vee (B \supset A) \mid \Rightarrow (A \supset B) \vee (B \supset A)}{\Rightarrow (A \supset B) \vee (B \supset A)}}$$

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Linearity

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow A}\,(\mathrm{com})}{\Rightarrow A \supset B \mid \Rightarrow B \supset A}}{\Rightarrow (A \supset B) \vee (B \supset A) \mid \Rightarrow (A \supset B) \vee (B \supset A)}}{\Rightarrow (A \supset B) \vee (B \supset A)}$$

History

Timeline

finitely valued logics

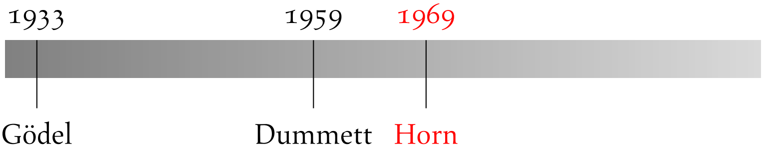
History

Timeline

infinitely valued propositional Gödel logics

History

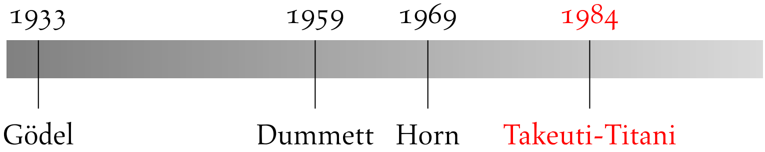
Timeline



linearly ordered Heyting algebras

History

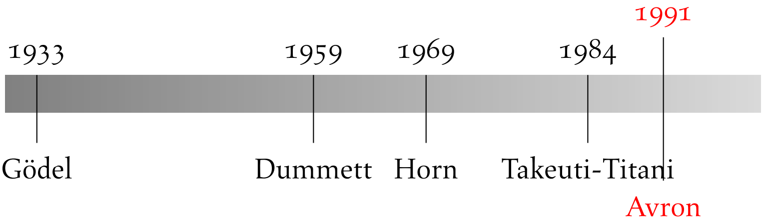
Timeline



intuitionistic fuzzy logic

History

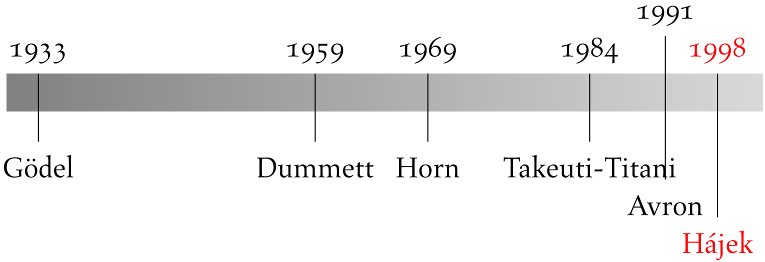
Timeline



hypersequent calculus

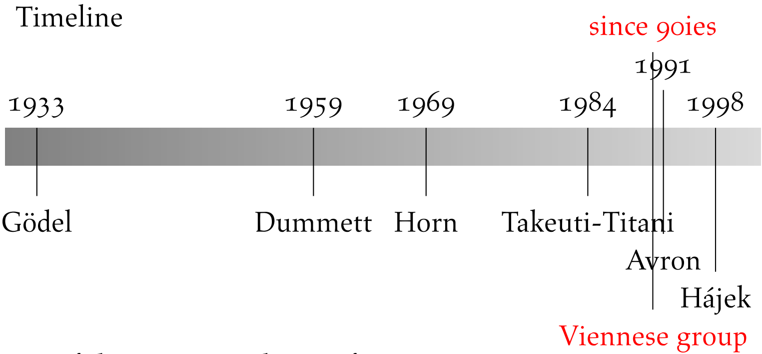
History

Timeline



t-norm based logics

History



proof theory, #, Kripke, qp, fragments, ...

Open problems

- intensional versus extensional definition
- Herbrand disjunctions
- Calculi for other than the standard logic
- equivalent of $L(\mathbb{R})$, the logic of the Kripke frame of $\mathbb{R}$ within an extended 'real' setting
- equivalence of 'one logic per truth-value set' for Gödel algebras
- quantified propositional logics – largely untapped
- computational model

Recapitulation

Recapitulation

Standard meta-theory

- soundness, completeness
- axiomatizability
- decidability of satisfiability an validity
- sub-classes, monadic and other fragments
- proof theory
- representation theorems

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Standard meta-theory

- soundness, completeness
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Relation to different areas

- order theory, topology, polish spaces
- Kripke frames
- (Heyting) algebras
- computation
- ...

Conclusion

Although not traditional logic, it provides a rich meta-theory and there are still many unexplored topics.

Application-wise of relevance due to ease of modelling and well-behaved logic even on first-order level. (medical expert system, database modelling, . . .)

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Fun!

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Fun!

Thanks