Gödel Logics – a short survey

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Today’s program

- Development of many-valued logics
t-norm based logics
- Gödel logics (propositionalquantified propositional, first order)
- Gödel logics and . . .
  - Topology
  - Order Theory
  - Computability
- Other topics
  - Kripke frames and beyond the reals
  - Monadic fragment
  - Proof theory
- History
- Conclusion
Development of many-valued logics

The most important stops

- **Platon, Aristotle (De Interpretatione IX), Ockham**: future possibilities, problem of determination vs. fatalism.

- **Łukasiewicz 1920**: 3-valued logic of non-determinism

- **Post 1920**: Many-valued logic dealing with functional completion

- **Gödel 1932**: Finite valued logics for approximation of intuitionistic logic

- **Böcvar 1938**: Logic of Paradoxa

- **Kleene 1952**: Logic of the unknown

- **Zadeh 1965**: Fuzzy sets and fuzzy logics
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How do we continue?

Arbitrary finite-valued logics

For all finite-valued logics with truth-value functions there is an automatic algorithm for generating a sequent calculus, proving completeness etc (MultLog, MultSeq: Baaz, Fermüller, Salzer, Zach et al. 1996ff).

Infinite valued logics

Does it make sense to take truth values from arbitrary partial orderings?

⇒ No, because every logics with substitution property would be a many-valued logic!

Take all sentences as truth values, and all sentences of the logic as designated truth values.
HOW DO WE CONTINUE?

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Design decisions

Basic requirements

- Extension of classical logic
- $[0, 1]$ as super-set of the truth value set
- functional relation between the truth value of a formula and the one of its sub-formulas.
**Design decisions**

**Basic requirements**
- Extension of classical logic
- \([0, 1]\) as super-set of the truth value set
- Functional relation between the truth value of a formula and the one of its sub-formulas.

**Additional ‘natural’ properties of the conjunction**
- Associative \(((A \land B) \land C \iff A \land (B \land C))\)
- Commutative \((A \land B \iff B \land A)\)
- Order preserving
  - If \(A\) is less true than \(b\), then \(A \land C\) is less (or equal) true than \(B \land C\).
- Continuous
Definition of (continuous) t-norms

Definition

A t-norm is an associative, commutative, and monotone mapping from $[0, 1]^2 \to [0, 1]$ with 1 as neutral element.

- $(x \star y) \star z = x \star (y \star z)$
- $x \star y = y \star x$
- $x \leq y \Rightarrow x \star z \leq y \star z$
- $1 \star x = x$
- $\star$ is continuous
**Definition of (continuous) t-norms**

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- $x \leq y \implies x \star z \leq y \star z$
- $1 \star x = x$
- $\star$ is continuous

**Algebraic view**

$\langle [0,1], \star, 1, \leq \rangle$ is a commutative and ordered monoid.
From t-norm to the logic

Every t-norm has a residuum

\[ x \star z \leq y \iff z \leq (x \Rightarrow y) \]

Truth functions for operators

▶ strong conjunction &: defined via the t-norm

▶ implication \( \supset \): defined via the residuum

▶ Negation: \( \neg A := A \supset \bot \)

▶ (weak) disjunction: \( A \lor B := (A \supset B) \supset B \)

▶ (weak) conjunction: \( A \land B := \neg (\neg A \lor \neg B) \)

▶ strong disjunction: \( A \disjoint B := \neg (A \supset \neg B) \)
From t-norm to the logic

The residuum of a t-norm

Every t-norm has a residuum

\[ x \star z \leq y \iff z \leq (x \Rightarrow y) \]

\[ x \Rightarrow y := \max\{z : x \star z \leq y\} \]
From t-norm to the logic

The residuum of a t-norm

Every t-norm has a residuum

\[ x \star z \leq y \iff z \leq (x \Rightarrow y) \]

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Truth functions for operators

- strong conjunction &: defined via the t-norm
- implication ⊃: defined via the residuum
- Negation: \( \neg A := A \supset \bot \)
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- (weak) conjunction: \( A \land B := \neg(\neg A \lor \neg B) \)
- strong disjunction: \( A \lor B := \neg(\neg A \supset \neg B) \)
Basic $t$-norms

Gödel

non-trivial idempotent elements
Basic $t$-norms

Gödel  
non-trivial idempotent elements

Łukasiewicz  
no non-trivial idempotent elements, but zero divisors
Basic t-norms

Gödel  Łukasiewicz  Product

- non-trivial idempotent elements
- no non-trivial idempotent elements, but zero divisors
- no non-trivial idempotent elements, no zero divisors
Representation of $t$-norm

Theorem (Mostert and Shields, 1957)

Every $t$-norm is the ordinal sum of Łukasiewicz $t$-norm and Product $t$-norm.
Questions and results

- Basic logic: the logic of all t-norms (Hajek 1998)
- Axiomatizability: propositional logic: easy, first-order: only Gödel logics are axiomatizable (Scarpellini 1962, Horn 1969, Takeuti, Titani 1984, Takano 1987)
- calculi for first order logic: only for Gödel logic (Baaz, Zach 2000)
- other questions: automatic theorem proving, size of families, …
Gödel Logics
Propositional Logics
**Propositional logic**

Usual propositional language, $\neg A$ is defined as $A \supset \bot$.

**Evaluations**

Fix a truth value set $\{0, 1\} \subseteq V \subseteq [0, 1]$
$v$ maps propositional variables to elements of $V$

$$v(A \land B) = \min\{v(A), v(B)\}$$
$$v(A \lor B) = \max\{v(A), v(B)\}$$
$$v(A \supset B) = \begin{cases} v(B) & \text{if } v(A) > v(B) \\ 1 & \text{if } v(A) \leq v(B). \end{cases}$$
Negation

This yields the following definition of the semantics of \( \neg \):

\[
\nu(\neg A) = \begin{cases} 
0 & \text{if } \nu(A) > 0 \\
1 & \text{otherwise}
\end{cases}
\]
Takeuti’s observation

Gödel implication

\[ \nu(A \supset B) = \begin{cases} \nu(B) & \text{if } \nu(A) > \nu(B) \\ 1 & \text{if } \nu(A) \leq \nu(B) \end{cases} \]

is the only one satisfying:

\begin{itemize}
  \item \( \nu(A) \leq \nu(B) \iff \nu(A \supset B) = 1 \)
  \item \( \Pi \cup \{A\} \vdash B \iff \Pi \vdash A \supset B \)
  \item \( \Pi \vdash B \Rightarrow \min\{\nu(A) : A \in \Pi\} \leq \nu(B) \)
    \( (\text{and if } \Pi = \emptyset \Rightarrow 1 \leq \nu(B)) \)
\end{itemize}
Definition of the logic

\[ G_V = \{ A : \forall v \text{ into } V : v(A) = 1 \} \]
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\[ G_V = \{ A : \forall v \text{ into } V : v(A) = 1 \} \]

Examples

\[ V = \{0, 1\} \quad \rightarrow \quad G_V = \text{CPL} \]
\[ V_1 = \{0, 1/2, 1\}, \quad V_2 = \{0, 1/3, 1\} \quad \rightarrow \quad G_{V_1} = G_{V_2} \]
\[ V_\uparrow = \{1 - 1/n : n \geq 1\} \cup \{1\} \quad \rightarrow \quad G_{V_\uparrow} = G_\uparrow \]
\[ V_\downarrow = \{1/n : n \geq 1\} \cup \{0\} \quad \rightarrow \quad G_{V_\downarrow} = G_\downarrow \]
Propositional completeness

- Lindenbaum algebra of the formulas
- show that the algebra $\mathcal{F}/\equiv$ is a subalgebra of

$$\mathcal{X} = \prod_{i=1}^{n!} \mathcal{C}(\bot, \pi_i(p_1, \ldots, p_n), \top)$$

($\mathcal{C}(\ldots)$ being the chain consisting of the listed elements, and the $\pi_i$ all the permutations) by defining

$$\phi(|\alpha|) = (|\alpha|_{c_1}, \ldots, |\alpha|_{e_{n!}})$$
Consequences

- countably many propositional Gödel logics

\[ G_2 \supset G_3 \supset \ldots \supset G_n \supset \ldots \supset G_\uparrow = G_\downarrow = G_V = G_\infty = \bigcap_{n \geq 2} G_n \]

(where \( V \) is any infinite truth value set)

- if \( f : V_1 \leftrightarrow V_2 \) with \( f(0) = 0 \) and \( f(1) = 1 \), order-preserving
  \( (x < y \Rightarrow f(x) < f(y)) \), then

\[ G_{V_1} \supseteq G_{V_2} \]

- check on satisfiability and validity
Quantified Propositional Logics
**Propositional Logic**

Fix a truth value set \( \{0, 1\} \subseteq V \subseteq [0, 1] \)
\( v \) maps propositional variables to elements of \( V \)

\[
\begin{align*}
v(A \land B) &= \min\{v(A), v(B)\} \\
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**Quantified Propositional Logic**

Fix a truth value set \( \{0, 1\} \subseteq V \subseteq [0, 1] \), \( V \) closed
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\[
v(\forall p A(p)) = \inf\{v(A(p)) : p \in P\}
\]
\[
v(\exists p A(p)) = \sup\{v(A(p)) : p \in P\}
\]
### Properties

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(Baaz, Veith, Zach, P. 2000–)
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- uncountably many different quantified propositional logics (coding the topological structure)
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- uncountably many different quantified propositional logics (coding the topological structure)
- $G^{qp}_{\uparrow} = \bigcap_{n \in \mathbb{N}} G^{qp}_n$
- $\bigcap_{V \subseteq [0,1]} G^{qp}_V$ is not a quantified propositional Gödel logic (in contrast to propositional and first-order Gödel logics)
First Order Logics
First Order Gödel Logics

Fix a truth value set \( \{0, 1\} \subseteq V \subseteq [0, 1] \), \( V \) closed

Interpretation \( \nu \) consists of

- a nonempty set \( U \), the universe of \( \nu \)
- for each \( k \)-ary predicate symbol \( P \) a function \( P^\nu : U^k \to V \)
- for each \( k \)-ary function symbol \( f \), a function \( f^\nu : U^k \to U \)
- for each variable \( x \) an object \( x^\nu \in U \)
Extend the valuation to all formulas

\[ \nu(A \land B) = \min\{\nu(A), \nu(B)\} \]

\[ \nu(A \lor B) = \max\{\nu(A), \nu(B)\} \]

\[ \nu(A \supset B) = \begin{cases} 
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\[ \nu(\forall x A(x)) = \inf\{\nu(A(u)) : u \in U\} \]

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HORN-TAKEUTI-TITANI-TAKANO – axiomatizability

Axiomatizability of $G_{[0,1]}$:  

LIN: \quad A \supset B \lor B \supset A  

QS: \quad \forall x (A(x) \lor B) \supset (\forall x A(x) \lor B)
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GL: $IL + LIN + QS$
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GL: IL + LIN + QS

- Horn (1969) logic with truth values in a linearly ordered Heyting algebra
- Takeuti-Titani (1984), Takano (1987) intuitionistic fuzzy logic
Takano’s proof

- set of formulas $\mathcal{F}$, equivalence relation $\equiv$ by provable equivalence
- show that $\mathcal{F}/\equiv$ is a (linear) Gödel algebra
- embed $\mathcal{F}/\equiv$ into $[0, 1]$
- show that embedding preserves infima and suprema (order-theoretic infima versus topological infima)
Connections

Gödel Logics and …

- topology
- order theory
- computation
Gödel Logics and Topology
Possible truth value sets

Perfect set

A set $P \subseteq \mathbb{R}$ is perfect if it is closed and all its points are limit points in $P$.

Cantor-Bendixon

Any closed $V \subseteq \mathbb{R}$ can be uniquely written as $V = P \cup C$, with $P$ a perfect subset of $V$ and $C$ countable and open.
Possible truth value sets

Perfect set

A set $P \subseteq \mathbb{R}$ is perfect if it is closed and all its points are limit points in $P$.

Cantor-Bendixon

Any closed $V \subseteq \mathbb{R}$ can be uniquely written as $V = P \cup C$, with $P$ a perfect subset of $V$ and $C$ countable and open.

Examples for perfect sets

- $[0, 1]$, any closed interval, any finite union of closed intervals
- Cantor Middle Third set $C$: all numbers of $[0, 1]$ that do not have a 1 in the triadic notation (cut out all open middle intervals recursively) (perfect but nowhere dense)
The $\triangle$ operator

\[ \triangle(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases} \]

- introduced and axiomatised by Takeuti and Titani in their discussion of intuitionistic fuzzy logic
- Baaz introduced and axiomatised in the context of Gödel logics
- parallels the ‘recognizability’ of 0, i.e., makes 1 recognizable.
- axiomatization of Gödel logics with $\triangle$ using Hilbert style calculus
**Axiomatisation of △**

Baaz gave the following Hilbert style axiomatisation of the △ operator:

1. △1 △A ∨ ¬ △A
2. △2 △(A ∨ B) ⊃ (△A ∨ △B)
3. △3 △A ⊃ A
4. △4 △A ⊃ △ △A
5. △5 △(A ⊃ B) ⊃ (△A ⊃ △B)
6. △R A ⊢ △A

Extension of the interpretation:

\[ ν(△A) = \begin{cases} 
1 & \text{if } ν(A) = 1 \\
0 & \text{if } ν(A) < 1 
\end{cases} \]
Full characterization of $\triangle$-Axiomatizability

Recursively axiomatizable

- finitely valued
- $V$ contains a perfect subset $P$ and for both $0$ and $1$ it holds that they are either in the perfect kernel or isolated (4 cases)
Full characterization of $\Delta$-Axiomatizability

Recursively axiomatizable

- finitely valued
- $V$ contains a perfect subset $P$ and for both 0 and 1 it holds that they are either in the perfect kernel or isolated (4 cases)

Not recursively enumerable

- countably infinite truth value set
- either 0 or 1 is not isolated but not in the perfect kernel
### Axiomatizability

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(Baaz, P., Zach 2007; Baaz, P. 2016)

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Gödel Logics and Order Theory
Dummett – number of different logics

Dummett (1959)

All propositional (Gödel) logics based on infinite truth value sets coincide. Thus, in total there are $\aleph_0$ different propositional logics.
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Quantified propositional

$\aleph_1$ by coding empty and non-empty intervals
**Dummett – number of different logics**

**Dummett (1959)**

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**Quantified propositional**

$\aleph_1$ by coding empty and non-empty intervals

**First order?**

Lower bounds: always $\aleph_0$ (finitely valued, quantifier alterations, Cantor-Bendixson rank)
Counting first order logics

Comparing logic

If there is an injective, continuous and order preserving embedding from $V_1$ into $V_2$ that preserves 0 and 1, then $G_{V_1} \supseteq G_{V_2}$.
Counting first order logics

Comparing logic

If there is an injective, continuous and order preserving embedding from $V_1$ into $V_2$ that preserves $0$ and $1$, then $G_{V_1} \supseteq G_{V_2}$.

Fraïssé Conjecture (1948), Laver (1971)

A $(Q, \leq)$ with reflexive and transitive $\leq$ is a quasi-ordering.

The set of scattered linear orderings ordered by embeddability is a well-quasi-ordering (does not contain infinite anti-chains nor infinitely descending chains)
Examples for quasi-orderings

Example

The collection of all linear orderings together with embeddability form a quasi-ordering, but not a partial ordering.

$\eta$ and $\eta + 1$ are different order types, but each embeddable into the other.
**Examples for quasi-orderings**

Example

The collection of all linear orderings together with embeddability form a quasi-ordering, but not a partial ordering.

$\eta$ and $\eta + 1$ are different order types, but each embeddable into the other.

Example

The collection of all linear orderings contain infinite descending chains, e.g. the order types of dense suborderings of $\mathbb{R}$. 
Transfer to Gödel logics

Generalized Fraïssé Conjecture

The class of countable closed subsets of the reals with respect to injective and continuous embeddability is a well-quasi-ordering.

Final result

The number of first order Gödel logics is $\aleph_0$.

(Beckmann, Goldstern, P., 2008)
Transfer to Gödel logics

Generalized Fraïssé Conjecture

The class of countable closed subsets of the reals with respect to injective and continuous embeddability is a well-quasi-ordering.

GFC for Gödel logics

The class of countable Gödel logics, ordered by $\supseteq$, is a wqo.
Transfer to Gödel logics

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The number of first order Gödel logics is $\aleph_0$.

(Beckmann, Goldstern, P. 2008)
Gödel Logics and Computation
Motivation

Motivation

  - The second, deeper objective of this paper is to contribute towards a better understanding of the notion of logical consequence in general, and especially its possible relations with parallel computations.
  - We believe that these logics [...] could serve as bases for parallel λ-calculi.
  - The name “communication rule” hints, of course, at a certain intuitive interpretation that we have of it as corresponding to the idea of exchanging information between two multiprocesses: [...]

Setting the stage

Every proof system hides a model of computation.

General aim: provide Curry-Howard style correspondences for parallel computation, starting from logical systems with good intuitive algebraic / relational semantics.
Every proof system hides a model of computation.
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Setting the stage

Every proof system hides a model of computation.

General aim: provide Curry-Howard style correspondences for parallel computation, starting from logical systems with good intuitive algebraic / relational semantics.
Wishlist

Properties we want to have:

(semi) local

- construction of deductions:
  apply ND inspired rules to extend a HND deductions
- modularity of deductions:
  reorder/restructure deductions
- analyticity (sub-formula property, ...)

(normalisation)

(procedural normalisation via conversion steps)
Wishlist

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normalisation

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Our approach to Hyper Natural Deduction
Our approach to Hyper Natural Deduction

\[(\text{com}) \quad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A} \quad \Gamma \quad \vdots \quad \Delta \quad \vdots \quad \Lambda\]
Our approach to Hyper Natural Deduction

\[
\begin{align*}
\text{(com)} \quad & \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A} \\
\quad & \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A} \\
\end{align*}
\]
Our approach to Hyper Natural Deduction

(com) \[
\frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}
\]

\[
\frac{\Gamma \quad \Delta}{\vdots \quad \vdots}
\]

\[
\frac{A \quad B}{com \ B}
\]
Our approach to Hyper Natural Deduction

\[
\begin{array}{c}
\text{(com)} \quad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}
\end{array}
\]
Our approach to Hyper Natural Deduction

\[
\begin{align*}
\text{(com)} & \quad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A} \\
\text{com} & \quad \frac{\Gamma}{\vdash \Delta} \\
& \quad \vdash \Delta \\
\text{com} & \quad \frac{A}{\vdash B} \\
& \quad \vdash \frac{B}{A}
\end{align*}
\]
Our approach to Hyper Natural Deduction

\[
\frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}
\]

\[
\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}
\]

- Consider sets of derivation trees
- Divide communication (and split) into two dual parts
- Search for minimal set of conditions that provides sound and complete deduction system
Reasoning in Hyper Natural Deduction

Double extension in *the spirit* of ND:

- from one tree to set of trees
- additional rules
Reasoning in Hyper Natural Deduction

Double extension in *the spirit* of ND:

- from one tree to set of trees
- additional rules

\[
\begin{align*}
&\text{From } \bar{A} \text{ and } B \text{ form } \text{com}_{A,B}^x \frac{A}{B} \quad \text{com}_{B,A}^x \frac{B}{A} \\
&\quad \vdots \quad \vdots \\
&\end{align*}
\]
**Reasoning in Hyper Natural Deduction**

Double extension in *the spirit* of ND:

- from one tree to set of trees
- additional rules

\[
\begin{array}{c}
\text{From } \bar{A} \text{ form } \com_{\bar{A},\overline{B}}^\chi \frac{\overline{A}}{B} \\
\text{From } \Gamma, \Delta \text{ form } \bar{x} : \text{Spt}_{\Gamma, \Delta}^\chi \frac{\bar{A}}{\bar{A}}
\end{array}
\]
Results for Hyper Natural Deduction

- sound and complete for standard first order Gödel logic
- procedural normalization
- sub-formula property

(Beatmann, P. 2016)
Results for Hyper Natural Deduction

- sound and complete for standard first order Gödel logic
- procedural normalization
- sub-formula property

(Beckmann, P. 2016)

Beauty of this system
- Hyper rules – derivations are completely in ND style
- Hyper rules mimic HLK/BCF system
- natural style of deduction
Other topics
Gödel Logics and Kripke Frames

Gödel logic to Kripke frame

For each Gödel logic there is a countable linear Kripke frame such that the respective logics coincide.

Kripke frames to Gödel logic

For each countable linear Kripke frame there is a Gödel truth value set such that the respective logics coincide.

(Beckmann, P. 2007)
**Going beyond $\mathbb{R}$**

**Takano (1987)**

Axiomatization of the logic of linear Kripke frames based on $\mathbb{Q}$ (which is that of $G_{[0,1]}$).

Axiomatization of the logic of linear Kripke frames based on $\mathbb{R}$ needs an additional axiom.
Monadic Fragment
## Decidability of validity and satisfiability

<table>
<thead>
<tr>
<th></th>
<th>validity</th>
<th>satisfiability</th>
</tr>
</thead>
<tbody>
<tr>
<td>finite $V$</td>
<td>full monadic</td>
<td>Yes</td>
</tr>
<tr>
<td>infinite $V$</td>
<td>full monadic</td>
<td>No</td>
</tr>
<tr>
<td>with $\Delta$</td>
<td>witnessed</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>quantifier prefix</td>
<td>$\forall^<em>\exists^</em>$</td>
</tr>
<tr>
<td>infinite $V$</td>
<td>full monadic</td>
<td>No</td>
</tr>
<tr>
<td>without $\Delta$</td>
<td>prenex</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>$\exists$, $\neg$-free</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>$\exists$</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>$\neg$-free</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>witnessed</td>
<td>No</td>
</tr>
</tbody>
</table>

(Baaz, Ciabattoni, P. 2011; Baaz, P. 2016)
Expressivity of Monadic logics

Take standard first order language.

Question: What can we express over complete linear orders?
Expressivity of Monadic logics

Take standard first order language.

Question: What can we express over complete linear orders?

Same question with one (1) monadic predicate symbol?
The results

Theorem

If $0 \prec \alpha \prec \beta \prec \omega^\omega$ with $\beta \geq \omega$, then $A_{\alpha, \beta} \in L(\alpha)$, but $A_{\alpha, \beta} \not\in L(\beta)$. 

(Beckmann, P 2014)
The results

Theorem

If $0 < \alpha < \beta < \omega^\omega$ with $\beta \geq \omega$, then $A_{\alpha,\beta} \in L(\alpha)$, but $A_{\alpha,\beta} \notin L(\beta)$.

Theorem

If $0 < \alpha < \beta < \omega^\omega$, then $A^*_\alpha \in L(\alpha^*)$, but $A^*_\alpha \notin L(\beta^*)$.

(Beckmann, P 2014)
Basic idea

Separate 2 from 3-valued logic

\[ (x_1 \supset x_2) \lor (x_2 \supset x_3) \]
Basic idea

Separate 2 from 3-valued logic

\[(x_1 \supset x_2) \lor (x_2 \supset x_3)\]
Proof theory

Hypersequent

Γ, Π finite multisets of formulas

Γ₁ ⇒ Π₁ | ... | Γₙ ⇒ Πₙ
Proof theory

Hypersequent

\[ \Gamma, \Pi \text{ finite multisets of formulas} \]

\[ \Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n \]

Rules

internal structural and logical (like LK)

external weakening and contraction

\[
\frac{G \mid \Gamma, \Delta \Rightarrow A \quad G' \mid \Gamma', \Delta' \Rightarrow A'}{G \mid G' \mid \Gamma, \Delta' \Rightarrow A \mid \Delta, \Gamma' \Rightarrow A'} \quad \text{(com)}
\]
Calculus HG

Sound and completeness

HG is sound and complete for Gödel logics (propositional and first order)
Calculus HG

Sound and completeness

HG is sound and complete for Gödel logics (propositional and first order)

Linearity

\[
\Rightarrow (A \supset B) \lor (B \supset A)
\]
Calculus HG

Sound and completeness

HG is sound and complete for Gödel logics (propositional and first order)

Linearity

\[ \Rightarrow (A \supset B) \lor (B \supset A) \lor (A \supset B) \lor (B \supset A) \]
Calculus HG

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Linearity

\[ \Rightarrow A \supset B | \Rightarrow B \supset A \]
\[ \Rightarrow (A \supset B) \lor (B \supset A) | \Rightarrow (A \supset B) \lor (B \supset A) \]
\[ \Rightarrow (A \supset B) \lor (B \supset A) \]
Calculus HG

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Linearity

\[
\frac{A \Rightarrow B \quad | \quad B \Rightarrow A}{\Rightarrow A \supset B \quad | \quad \Rightarrow B \supset A}
\]

\[
\Rightarrow (A \supset B) \lor (B \supset A) \quad | \quad \Rightarrow (A \supset B) \lor (B \supset A)
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Calculus HG

Sound and completeness

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Linearity

\[
\frac{A \Rightarrow A \quad B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow A} \quad \text{(com)}
\]

\[
\Rightarrow A \supset B \mid \Rightarrow B \supset A
\]

\[
\Rightarrow (A \supset B) \lor (B \supset A) \mid \Rightarrow (A \supset B) \lor (B \supset A)
\]

\[
\Rightarrow (A \supset B) \lor (B \supset A)
\]
History

Timeline

1933

Gödel

finitely valued logics
History

Timeline

1933  1959

Gödel  Dummett

infinitely valued propositional Gödel logics
linearly ordered Heyting algebras
intuitionistic fuzzy logic
History

Timeline

Gödel Dummett Horn Takeuti-Titani Avron

hypersequent calculus
History

Timeline


Gödel  Dummett  Horn  Takeuti-Titani  Avron  Hájek

t-norm based logics
History

Timeline

1933
Gödel

1959
Dummett

1969
Horn

1984
Takeuti-Titani

1991
Avron

1998
Hájek

since 90ies

Viennese group

proof theory, #, Kripke, qp, fragments, …
Open problems

- intensional versus extensional definition
- Herbrand disjunctions
- Calculi for other than the standard logic
- equivalent of $L(\mathbb{R})$, the logic of the Kripke frame of $\mathbb{R}$ within an extended ‘real’ setting
- equivalence of ‘one logic per truth-value set’ for Gödel algebras
- quantified propositional logics – largely untapped
- computational model
Recapitulation
Recapitulation

Standard meta-theory

- soundness, completeness
- axiomatizability
- decidability of satisfiability and validity
- sub-classes, monadic and other fragments
- proof theory
- representation theorems
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Standard meta-theory
- soundness, completeness
- axiomatizability
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- sub-classes, monadic and other fragments
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- representation theorems

Relation to different areas
- order theory, topology, polish spaces
- Kripke frames
- (Heyting) algebras
- computation
- …
Conclusion

Although not traditional logic, it provides a rich meta-theory and there are still many unexplored topics.

Application-wise of relevance due to ease of modelling and well-behaved logic even on first-order level. (medical expert system, database modelling, . . .)
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Thanks