L211 Logic and Mathematics

12. Lecture

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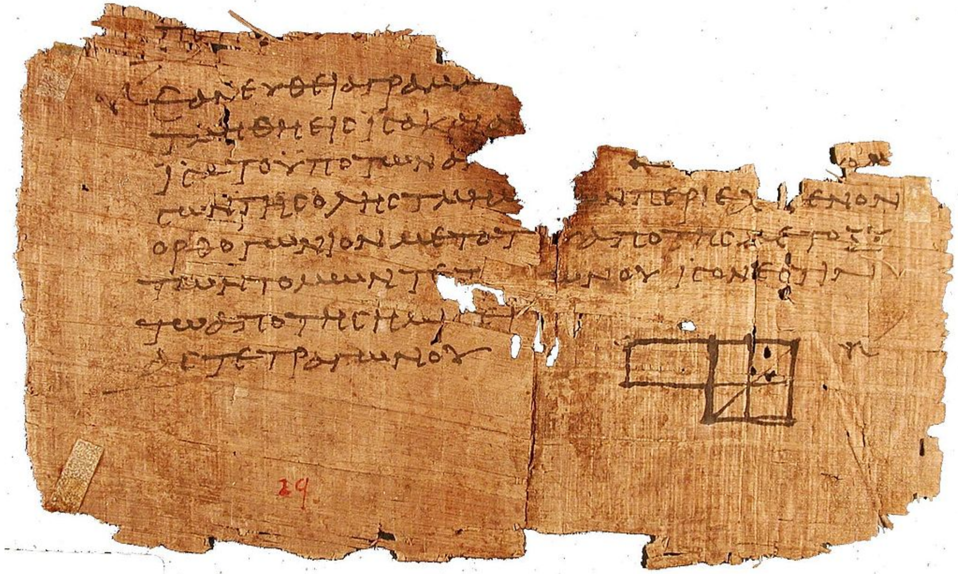
Axiom system

- undefined concepts
  Example: Point, line, circle, angle
- Axioms
  Example: ...
- Inference rules
  Example: For two abitrary distinct points, one can construct the line through these points.

Euclid(es) of Alexandria

- Greek mathematician
- 323–283BC
- perspective drawings,
  conic curves,
  astronomy, cartography
- Euclid's Elements



Euclid's Elements

- 13 volumes
- Content of vol. 1-4: development of geometry:
  - basic concept: point, line
  - set of axioms, axiom system
  - about 500 theorems with proofs

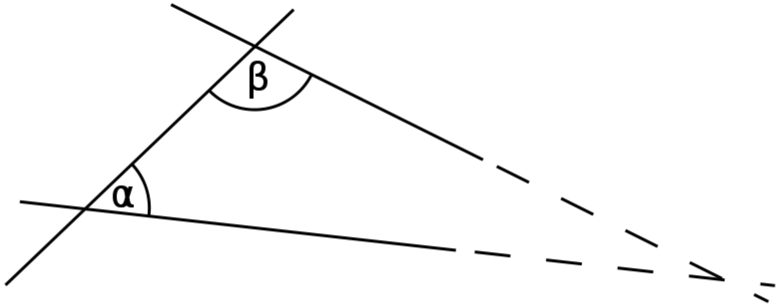
Euclid's axioms

Let the following be postulated:

- To draw a straight line from any point to any point.
- To extend a finite straight line continuously in a straight line.
- To draw a circle with any center and radius.
- That all right angles are equal to each other.
- Parallel postulate

Parallel Postulate

That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.



Equivalent properties

- There is at most one line that can be drawn parallel to another given one through an external point.bis
- The sum of the angles in every triangle is 180°.
- The sum of the angles is the same for every triangle.
- If three angles of a quadrilateral are right angles, then the fourth angle is also a right angle.

History of the Parallel Postulate

- a bit different axiom... many wrong 'proofs'
- Ptolemaios, c.90–c.168
- Proclus Lycaeus, 412–485
- Ibn al-Haitham, 965–1040
- Omar Khayyám, 1048–1131
- Nasir al-Din al-Tusi, 1201-12
- Giordano Vitale, 1633-1711
- Johann Lambert, 1728–1777
- ...

Finally in the 19th century

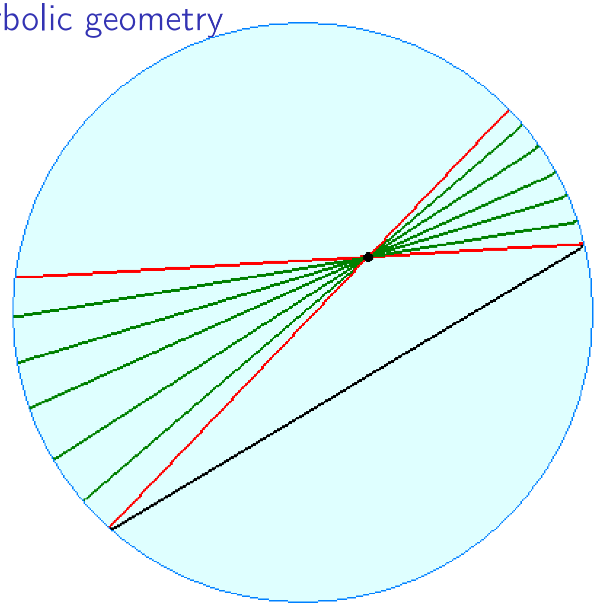
- Nikolai Ivanovich Lobachevsky, 1792–1856
- János Bolyai, 1802–1860
- Bernhard Riemann, 1826–1866

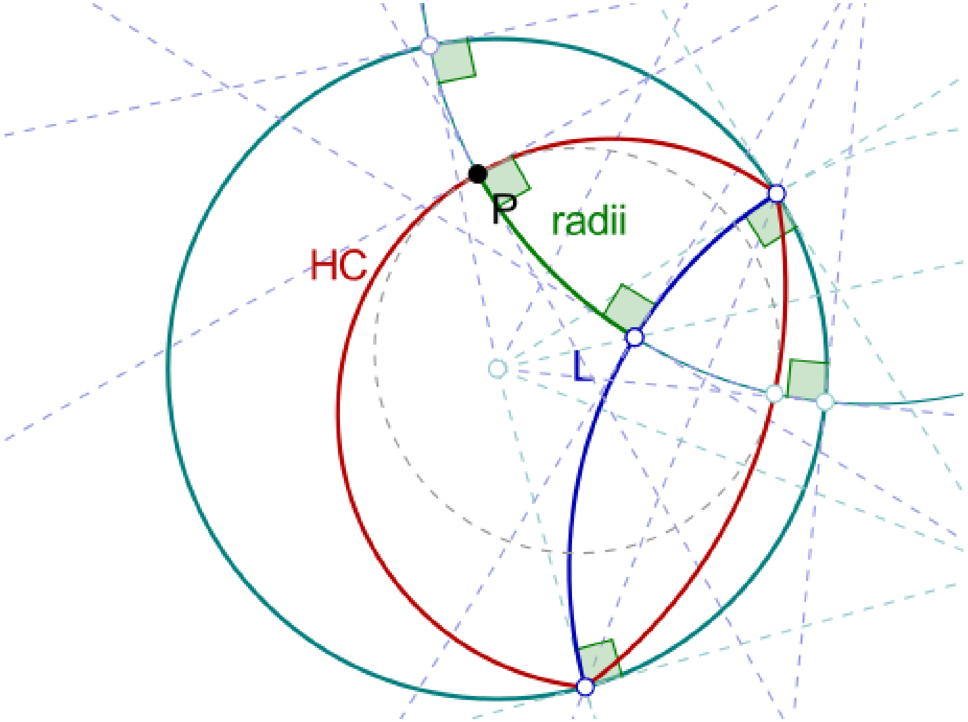
Rejection of the Parallel Postulate

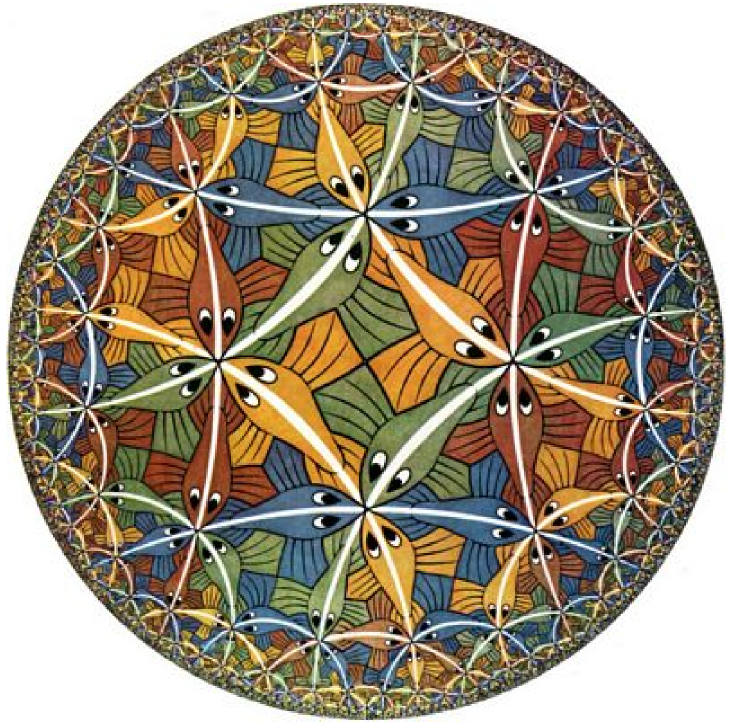
- Non-Euclidean Geometry
- Geometries were the Parallel Postulate does not hold

	Riemann	Euclid	Lobachevsky
Number of parallels	0	1	2
Shape	elliptic (spheric)	flat	hyperbolic (saddle)

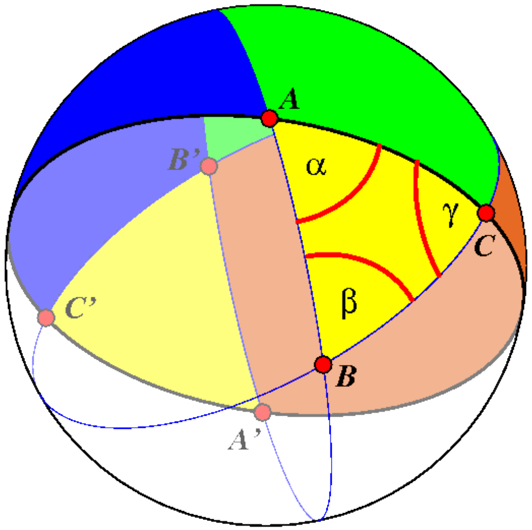
Hyperbolic geometry

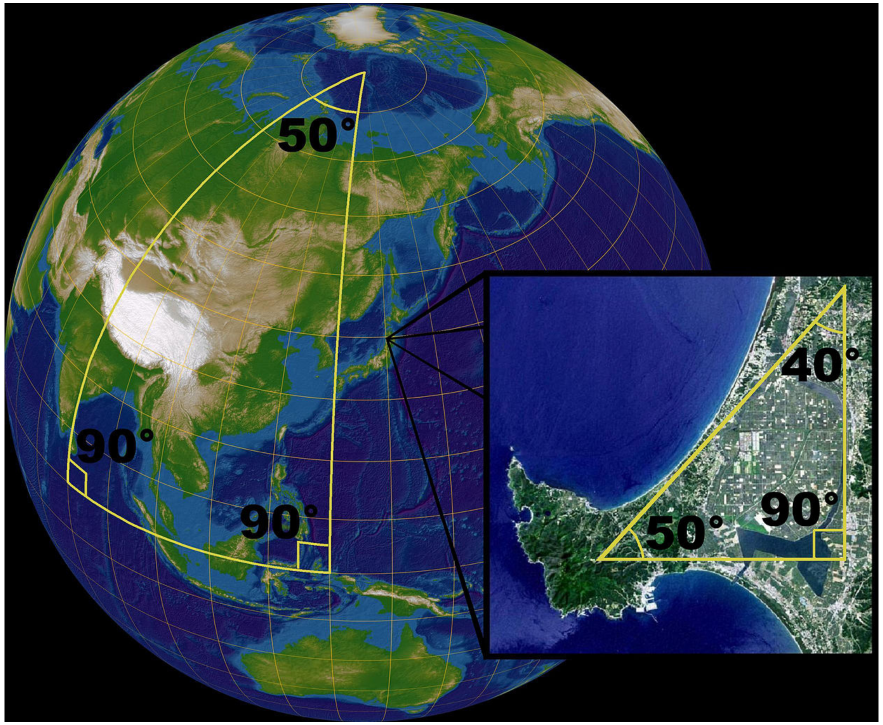






Elliptic geometry





Function and Limits

What is a function?

- $x^2 + 3x - 4$?
- $f(x) = x^2 + 3x - 4$?
- $y = f(x)$?
- $y = f(x)$ where $x^2 + y^2 = 1$?
- $f(n) = |\{p : p \in \mathbb{P} \land p \leq n\}|$?
- ???

Specifics of a function

- independent variable (e.g., x, n, $\ldots$)
- set of values for the independent variable: domain
- dependent variable (e.g., y, m, $\ldots$)
- values for the dependent variable: co-domain, range
- graph of the function: $\{(x, f(x)) : x \in D\}$

Examples for functions

$m = 2n$

- independent variable: n
- domain: $\mathbb{N}$
- dependent variable: m
- range: $2\mathbb{N}$ ($\mathbb{N}$)
- graph: $\{(n, 2n) : n \in \mathbb{N}\}$

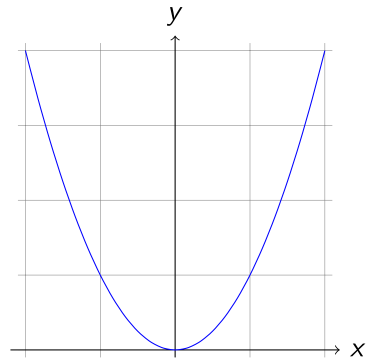
Examples of functions

$$S_n = S(n) = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n(n+1)(2n+1)}{6}$$

- independent variable: n
- domain: $\mathbb{N} \setminus \{0\}$
- dependent variable:(S)
- range: $\subset \mathbb{N}$
- graph: $\{(n, \frac{n(n+1)(2n+1)}{6}) : n > 0\}$

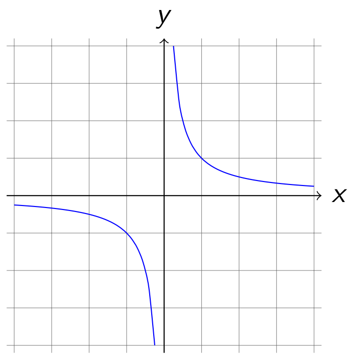
Example: $y = x^2$

- independent variable: x, domain: $\mathbb{R}$
- dependent variable: y, rangeL $\mathbb{R}^{\geq 0}$
- graph: $\{(x, x^2) : x \in \mathbb{R}\}$



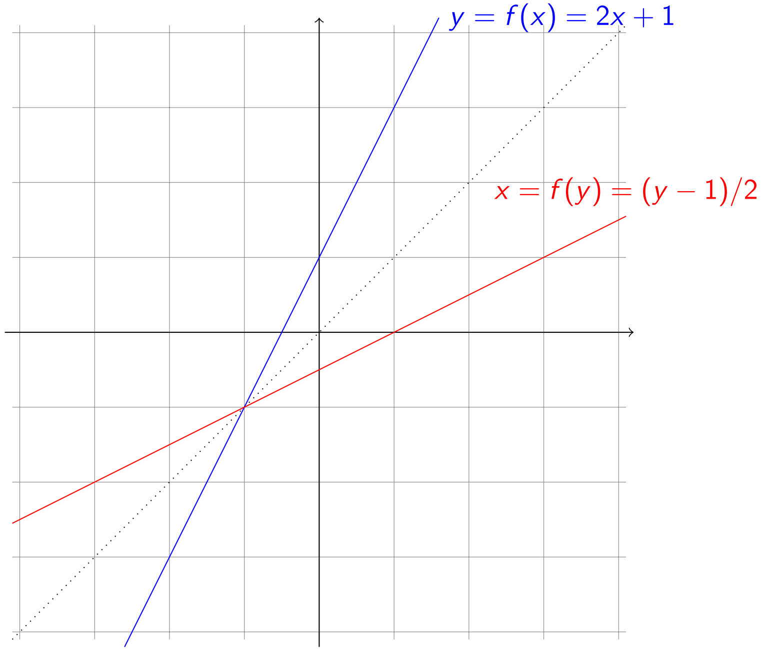
Example:: $f(x) = \frac{1}{x}$

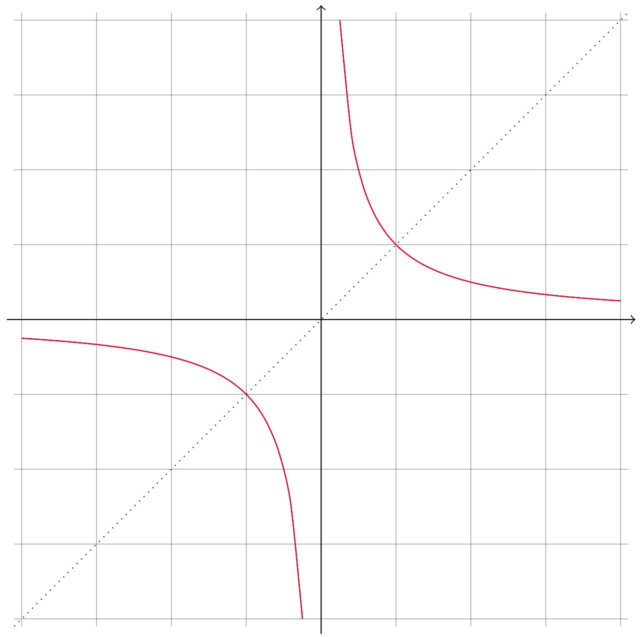
- independent variable: x, domain: $\mathbb{R} \setminus \{0\}$
- dependent variable: $(f(x))$, range: $\mathbb{R} \setminus \{0\}$
- graph: $\{(x, \frac{1}{x}) : x \in \mathbb{R} \setminus \{0\}\}$

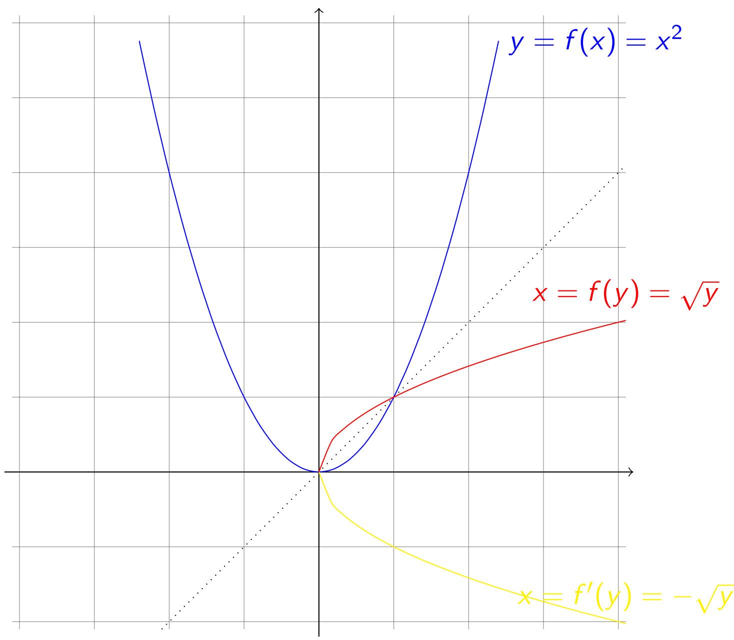


Inverse function

- $y = 2x + 1$
- independent variable $\leftrightarrow$ dependent variable
- $x = \frac{y-1}{2}$
- domain $\leftrightarrow$ range
- graph: reflection on $y = x$







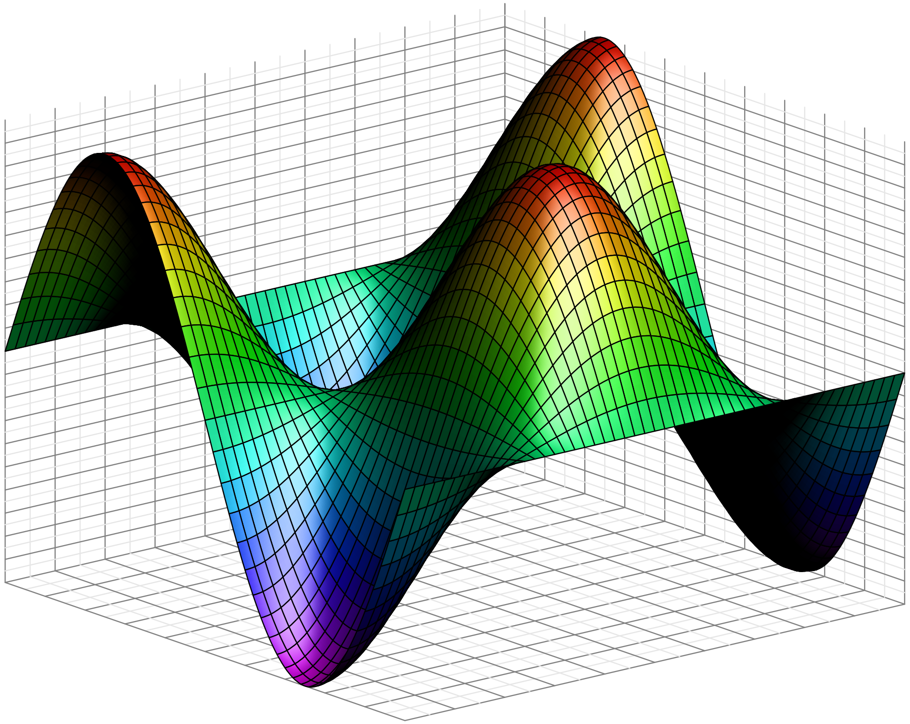
Inverse function cont.

The inverse function exists if $\forall y \exists! x \; f(x) = y$ holds.

$y = x^2 : \; 3^2 = 9 = (-3)^2$

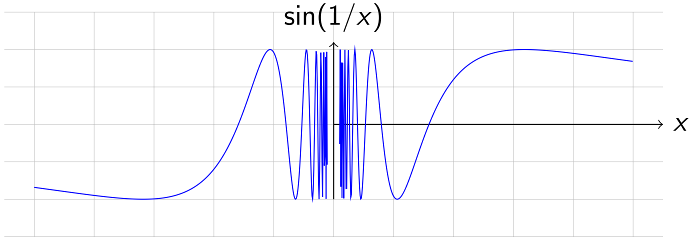
Functions with multiple variables

- independent variables: (e.g., x, y)
- domain: (e.g., $\mathbb{R} \times \mathbb{R}$)
- dependent variable (e.g., z)
- range: (e.g., $\mathbb{R}$)
- graph (e.g., $\{(x, y, z) : z = \cos(2\pi x) \times \sin(2\pi y)\}$)



$f(x) = \sin \frac{1}{x}$

- independent function: x, domain: $\mathbb{R} \setminus \{0\}$
- dependent function: $(f(x))$, range: $[-1, 1]$
- graph: $\{(x, \sin \frac{1}{x}) : x \in \mathbb{R} \setminus \{0\}\}$



$x \to 0$???

Sequence and rational numbers

- (repetition) Between any two different rational numbers x and y, there is another rational number. – denseness, dense
- Is the limit if any sequence of rational numbers again rational?

$$3 < 3.1 < 3.14 < 3.141 < \cdots < \pi < \cdots < 3.142 < 3.15 < 3.2 < 4$$

Sequences, limits, convergence

Sequence

$\langle a_0, a_1, a_2, \dots \rangle$

A function with $\mathbb{N}$ as domain: $a_n = a(n)$

Convergence

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \dots, \tfrac{1}{n}, \dots$$

If n grows infinitely big, $\frac{1}{n}$ will approach 0

$$\text{Notation: } \lim_{n\to\infty} \frac{1}{n} = 0$$

The meaning of convergence

- $a_n \to a$: a_n eventually comes arbitrary close to a.
- However small a neighborhood of a we choose, from some point onward all a_n fall into this nbh.
- $a_n = \frac{1}{n}$
  If the nbh is $(-0.1, 0.1)$, then $n > 10$
  If the nbh is $(-0.01, 0.01)$, then $n > 100$
  If the nbh is $(-0.001, 0.001)$, then $n > 1000$
  ...

Definition of convergence

For all $\varepsilon > 0$, there is a natural number N, such that for all $n > N$, $|a_n - a| < \varepsilon$

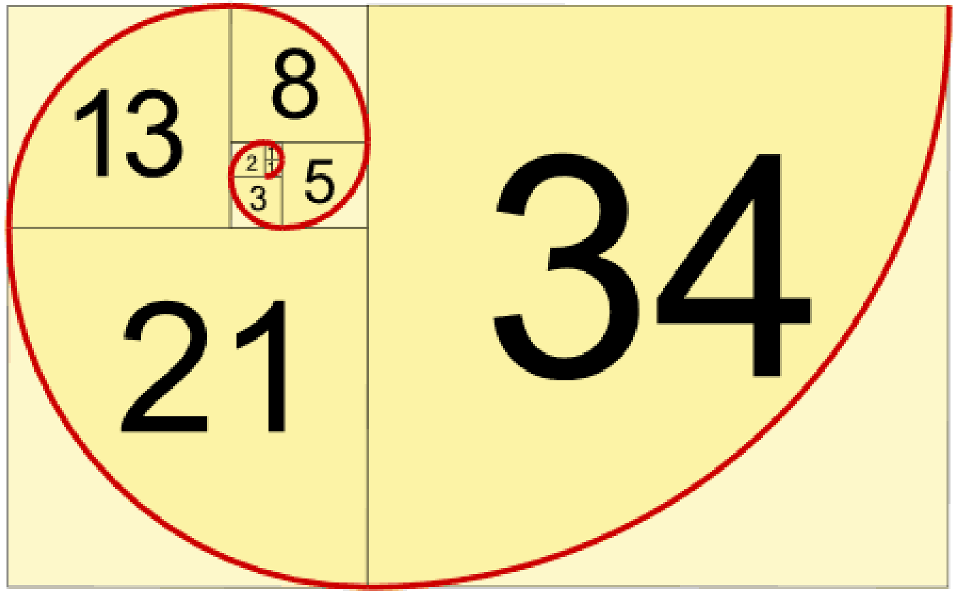
$$\forall \varepsilon > 0\, \exists N\, \forall n > N\, |a_n - a| < \varepsilon$$

Karl Weierstraß

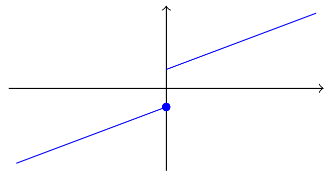
- German mathematician
- 1815–1897

Examples of converging sequences

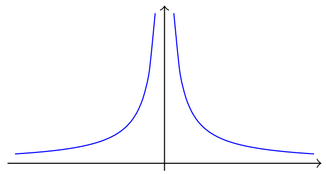
- $\lim_{n\to\infty} 1 - \frac{1}{n}$
- $\lim_{n\to\infty} \frac{2n+3}{5n}$
- $\lim_{n\to\infty} \frac{1-2q^n}{1+q^n} \quad (q \neq -1)$
- Fibonacci numbers: $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_n + F_{n+1}$
  0, 1, 1, 2, 3, 5, 8, 13, ...
  $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2} = 1.618\dots$ (Golden ratio)



Limits and continuity



$f(x) = 1 + x/2 \qquad x > 0$
$f(x) = -1 + x/2 \qquad x \leq 0$



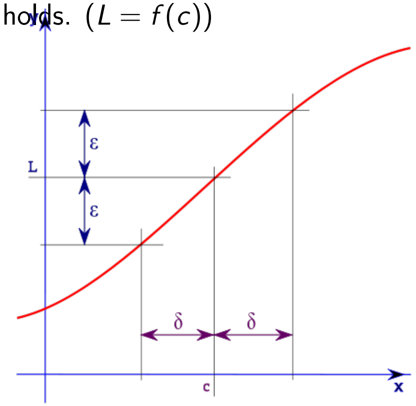
$f(x) = \frac{1}{x^2}$

- $\lim\limits_{x \to a} f(x)$
- $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = ? \qquad \cos x < \frac{\sin x}{x} < 1, \qquad 0 < x < \frac{\pi}{2}$

Continuous functions

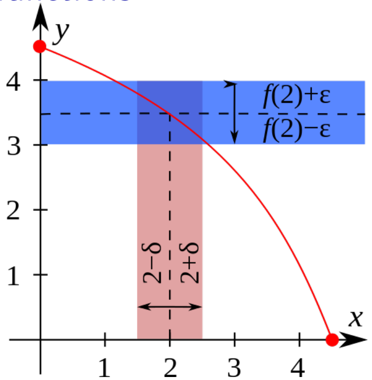
f is continuous in c iff

for any arbitrary positive number ε, there is a positive number δ, such that for all x such that $0 < |x - c| < \delta$ holds, also $|f(x) - L| < \varepsilon$ holds. $(L = f(c))$



Example of a continuous functions

$f(x) = x^3$, $c = 2$



Convergence and continuity

Convergence

$$\forall \varepsilon > 0 \, \exists N \, \forall n > N \, |a_n - a| < \varepsilon$$

Continuity

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in (c - \delta, c + \delta) \, |f(x) - L| < \varepsilon$$

ε-δ definitions

Series

For a sequence of numbers $a_1, a_2, \ldots$, let

$$s_n = a_1 + a_2 + \cdots + a_n$$

Example: $a_n = \frac{1}{10^n}$

$$\begin{aligned} s_1 &= a_1 && = 0.1 \\ s_2 &= a_1 + a_2 && = 0.11 \\ s_3 &= a_1 + a_2 + a_3 && = 0.111 \\ s_n &= a_1 + a_2 + a_3 + \ldots + a_n && = 0.1111\ldots1 \end{aligned}$$

$$\lim_{n \to \infty} s_n$$

Various answers

- $1 - 1 + 1 - 1 + 1 - 1 + \dots$
  - Answer 1: $(1-1)+(1-1)+(1-1)+\dots = 0$
  - Answer 2: $1-(1-1)-(1-1)-(1-1)-\dots = 1$
  - Answer 3: $1+1-1+1-1+1-1+\dots = \dots = 2$
- $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \to \infty$
- $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots \to 1$
- $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \to \frac{\pi^2}{6}$

Series and functions

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \quad (|x| < 1)$$

Power series

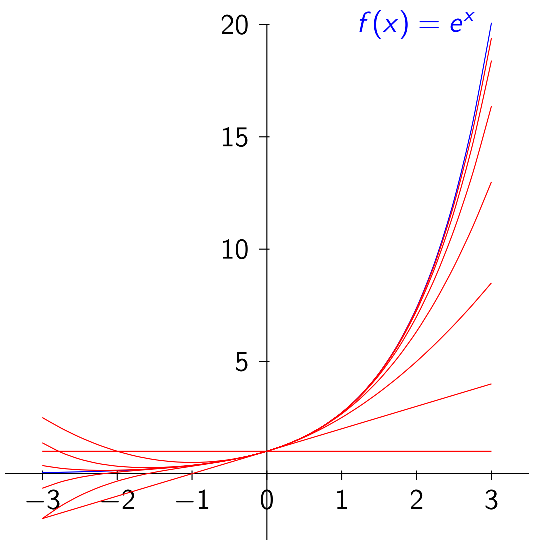
$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots \end{aligned}$$

Example for a power series

$$\begin{aligned} \sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \end{aligned}$$

$$\begin{aligned} \cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$



$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}$

How can we find the power series?

Development of power series, Taylor series

Oscillating differentiable function $f(x)$

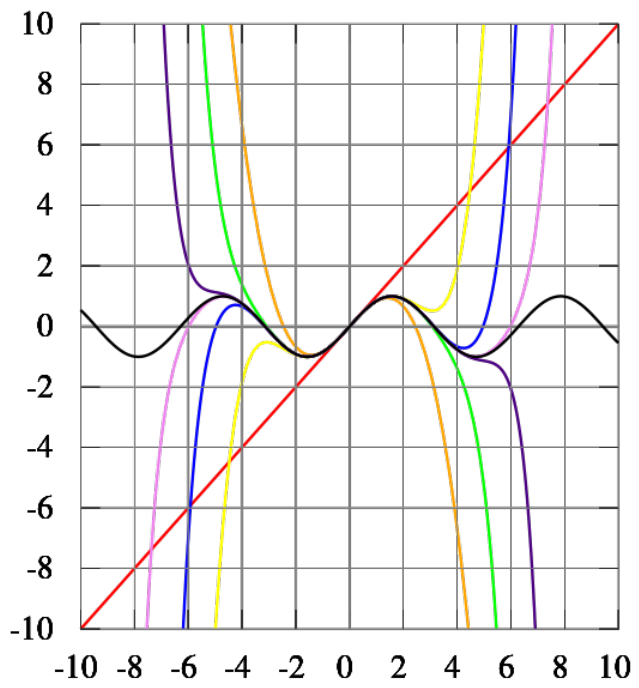
$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Differentiation

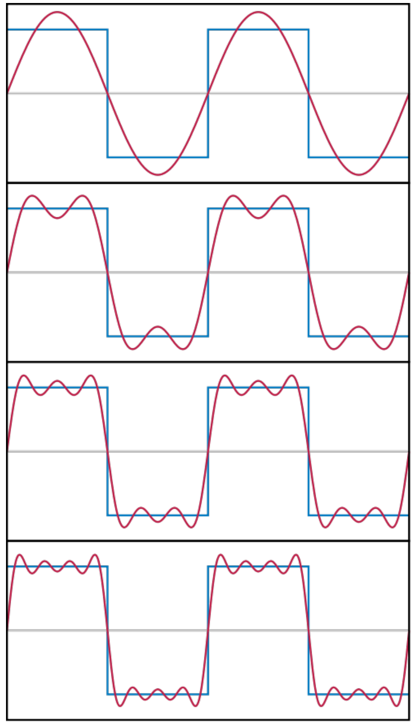
If
$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
exists, then $f(x)$ is differentiabl at a.

- $f(x) = 3x + 1$?
- $f(x) = x^3 - 5x^2 + 3x - 1$?
- $f(x) = |x|$?



Conditions on the existence of power series

- For all n, the differential $f^{(n)}(a)$ exists.
- $\lim_{n\to\infty} \frac{f^{(n)}(\theta x)}{n!} x^n = 0, \ (0 < \theta < 1)$

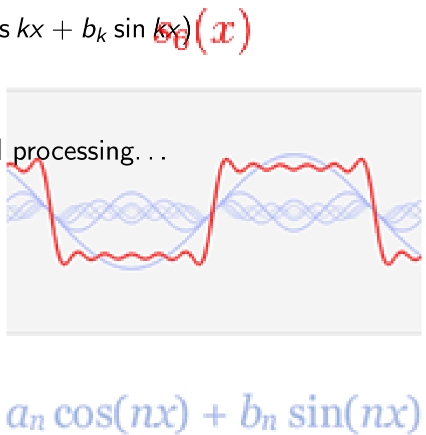


Fourier series

Periodic function

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

Electronic, accustic, optic, signal processing...



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Fibonacci spiral: `http://www.mathsisfun.com/`