L211 Logic and Mathematics

11. Lecture

Norbert PREINING
preining@jaist.ac.jp

http://www.preining.info/jaist/l211/2015e/

Axiomatic method

Formalization of number theory

- 0 is the first natural number.
- If x is a natural number, then also $s(x)$.
- If x and y are natural numbers, so are $x + y$ and $x \times y$.

Does this really give $\mathbb{N}$?

Condition: All natural numbers are only constructed from 0 and $s(x)$.

Mathematical induction
$[0 \in A \wedge \forall x(x \in A \to s(x) \in A)] \to \forall x(x \in \mathbb{N} \to x \in A)$

Peano axioms

Giuseppe Peano

- Italian mathematician
- 1858–1932
- Russel — Peano

Peano axioms

- $s(x) \neq 0$, $s(x) = s(y) \rightarrow x = y$
- $x + 0 = x$, $x + s(y) = s(x + y)$
- $x \times 0 = 0$, $x \times s(y) = (x \times y) + x$
- $[\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(s(x)))] \rightarrow \forall x \phi(x)$

Question: Is the Peano axiom system free of contradictions (consistent)? — And how to prove it?

Consistent means we *cannot* prove $0 = 1$.

Is it ok to use set theory to prove consistency? — no!

Fundamental theorems

- Gödel's completeness theorem, 1929
  Provability and semantical truth of first order predicate logic coincides.
- Church's thesis, 1936
  Every computable function is recursive.
  There is no algorithm to decide the truth of an arbitrary statement of first order predicate logic.

Peano axiom system – Gödels Incompleteness theorem

Gödel's Incompleteness Theorem

> *For any ω-consistent theory that includes the natural numbers and induction, there are true statements that can neither proved nor disproved.*

Condition: ω-consistent

Consequence

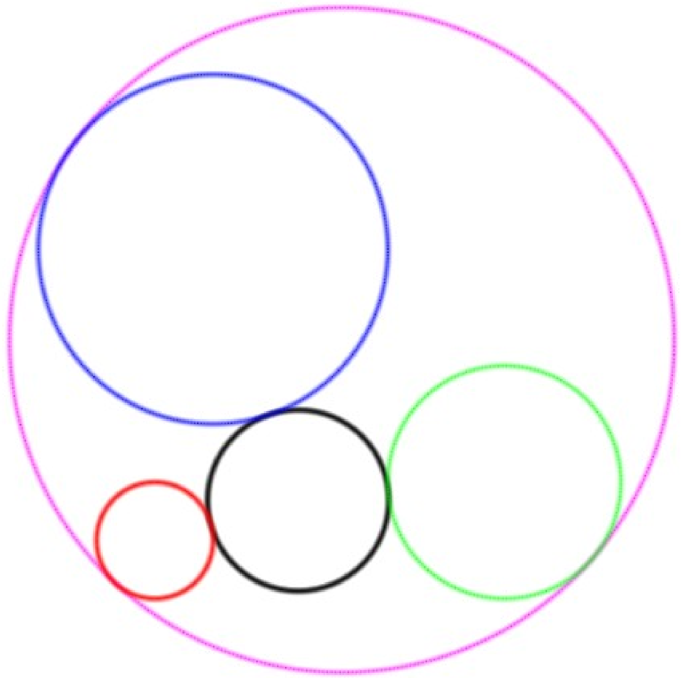
In Peano's system there are statements that are true but cannot be proved (nor disproved).

Geometry as axiom systems

Non-Euclidean Geometry

Axiom system

- Undefined concepts
  Example: Peano: $0, s$
- Axioms
  Example: Peano: $s(x) \neq 0$
- Inference rules
  Example: Peano: Induction



Constructions with compass and ruler (straightedge)

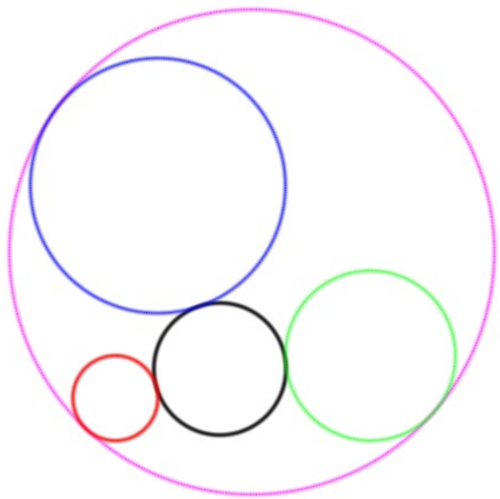
- ruler: connecting two points with a straight line (no measuring!)
- compass: draw a circle around a center

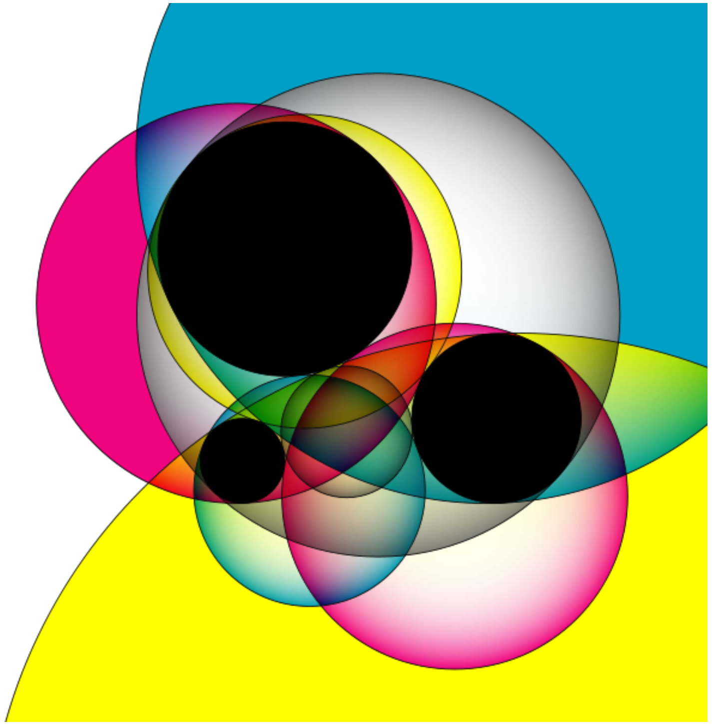
Basic constructions: 'axioms'

- For two given different points, draw the straight line through these points.
- For two given different points, use one as center and draw a circle through the other.
- For two non-parallel lines, determine the intersection point.
- For a given circle and a straight line, determine the intersection points (at most 2).
- For two given points, determine the intersection points (at

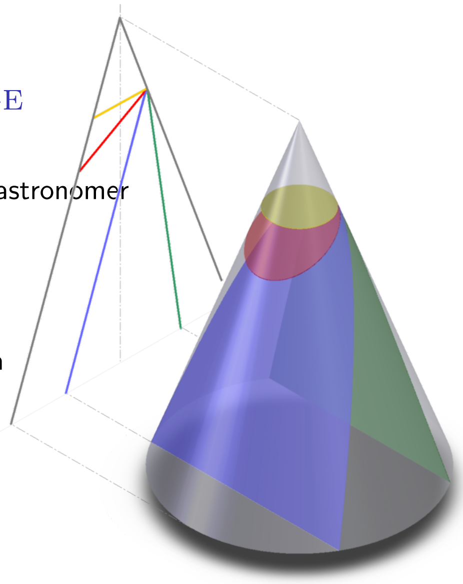
What can be con

- straight line
- center point
- half of an angle
- polygons: for which n?
- Problem of Apollonius





Apollonios of Perge



- Greek mathematician and astronomer
- 262BC–190BC
- conic curves
  ellispe, parabola, hyperbola

Questions, questions, questions

Problems couldn't be constructed since the Greek times:

- Construct a square with the same area as a given circle.
- Construct a cube with double the volume of a given cube.
- Trisect a given angle.

Karl Friedrich Gauss

- Germany, 1777–1855
- Mathematician, astronomer, physicist
- Princeps mathematicorum

Gauss and the polygons

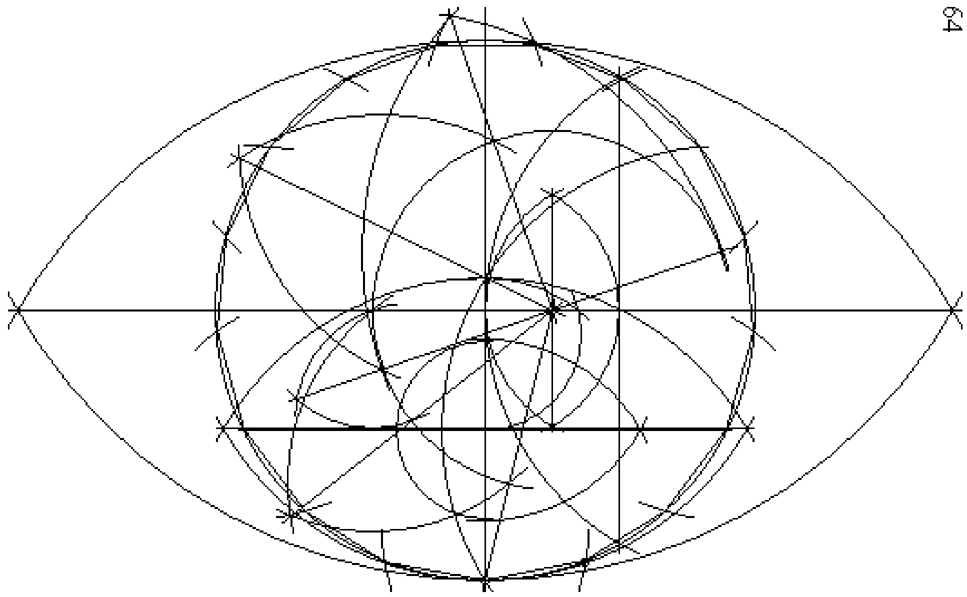
In the 1796, at the age of 19

A regular n-Polygon can be constructed with compass and ruler only

$$\Updownarrow$$

$$n = 2^k p_1 p_2 \dots p_t, \quad k, t > 0,\ p_i \text{ Fermat prime}$$

- Fermat primes: $F_n = 2^{(2^n)} + 1$ (3, 5, 17, 257)
- $F_5, \dots, F_{32}$ is not a real prime!
- $n = 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, \dots$ can be constructed



Sources

Wikimedia, partly CC BY 3.0 (Appollonius problem, user WillowW)