L211 Logic and Mathematics

10. Lecture

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Set theory

WHAT IS A SET?

A set is a collection of things.

1870ies: Cantor and Dedekind:

- Dedekind cut, reals
- ► Cantor: cardinality

Basics of abstract mathematics

- ► From the concrete sets N, Q to the abstract concepts 'group', 'ring', 'field'
- ▶ function space...

A QUICK PRIMER IN SET THEORY

- $x \in A$: means that 'x is a member of A'
- Axiom of Extensionality:

If A and B have the same members, they are the same set.

 $\forall A \forall B (\forall x (x \in A \leftrightarrow x \in B) \rightarrow A = B)$

This is not a definition, but a fundamental property of sets.

Example: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

ZERMELO-FRÄNKEL AXIOM SYSTEM

- Axiom of extensionality: $\forall A \forall B (\forall x (x \in A \leftrightarrow x \in B) \rightarrow A = B)$
- ► Axiom of empty set: $\exists A \forall x (x \notin A)$
- Axiom of pairing: $\forall x \forall y \exists A \forall t (t \in A \leftrightarrow (t = x \lor t = y))$
- ► Axiom of union $\forall X \exists A \forall t (t \in A \leftrightarrow \exists x \in X(t \in x))$
- Axiom of infinity $\exists A(\emptyset \in A \land \forall x \in A(x \cup \{x\} \in A))$
- Axiom of power set $\forall X \exists A \forall t (t \in A \leftrightarrow t \subseteq X))$

IS ALL OF MATHEMATICS NOTHING BUT SET THEORY?

One can give models of 'all' mathematics based on set theory:

- \blacktriangleright using the axioms of empty set and union, one can construct $\mathbb N$
- \blacktriangleright using $\mathbb N$ and infinity axiom, one can construct $\mathbb Q$
- \blacktriangleright using Dedekind cut (again sets) one can construct ${\mathbb R}$
- using pair axiom again, one can construct a function space
- ► ...

COMPREHENSION AXIOM

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- ► For all propositions P, there is a set of all x such that P(x) holds.
- For all propositions P, $\{x : P(x)\}$ exists.

PARADOXA IN MATHEMATICAL LOGIC

- Liers paradox: 'This sentence is wrong!'
- Barber's paradox

The barber who shaves exactly all those men who do not shave their own beards.

Who shaves the barber's beard?

If one proceeds like that, paradoxa appear ...

GEORG FRIEDRICH HEGEL

Georg Friedrich Hegel

- German philosopher
- ▶ 1770-1831

What experience and history teaches us is that people and governments have never learned anything from history, or acted on principles deduced from it.

Was die Erfahrung aber und die Geschichte lehren, ist dieses, dass Völker und Regierungen niemals etwas aus der Geschichte gelernt und nach Lehren, die aus derselben zu ziehen gewesen wären, gehandelt haben.



BERTRAND RUSSEL

- British philospher, logician, mathematician, 1872–1970
- Principia Mathematica



RUSSEL'S PARADOX

- Comprehension axiom: For all P, the set $\{x : P(x)\}$ exists.
- ▶ In the above axiom, use for P(x) the proposition $x \notin x$
- Thus, the set $a := \{x : x \notin x\}$ exists.
- Thus, for all $x, x \in a \Leftrightarrow x \notin x$
- Now set x = a, we obtain:

 $a \in a \Leftrightarrow a \notin a$

WAYS TO REPAIR THE SYSTEM

Restricted comprehension axiom If C is an existing set, then for all propositions P, the set $\{x \in C : P(x)\}$ exists.

Can you verify that in this case Russel's paradox does not occur?

Contradiction!

Other paradoxa?

- Using the restricted comprehension axiom we can pass by Russel's paradoxon.
- But do *all* paradoxa disappear?
- How can one guarantee contradiction freeness?
- ▶ What is the difference of truth in mathematics and physics?

Foundations of Mathematics

The method of Formalism

- One can construct mathematical logic independent of any relation to the outside world.
- There is only one requirement: contradiction-free logic
- ► How to deal with Paradoxa?
- A proof for the contradiction-freeness is necessary

Foundations of Mathematics

- ▶ Formalism: Hilbert, x, \forall , \exists , =, \leftrightarrow , undefined concepts
- Intuitionism: Brouwer, no infinite
- ► Logicism: Russel

PROOF OF CONSISTENCY (CONTRADICTION-FREENESS)

A logical theory *T* is *consistent*

 \uparrow iff

one cannot give a prove of a contradictory statement.

(Example: $a \land \neg a$)

- To show that a statement a is provable, we only need to provide a proof.
- But to show that a statement a is not provable, how would we proceed here?
 How can we check an infinite amount of proofs?



FORMALIZATION OF A THEORY

- If one plays Kb1, then white will win.
- To give a proof, we need to know the rules and clearly state them.
- To prove consistency, we need to know and clearly state the rules of logic!
- ► Formalization of mathematical logic: Hilbert, Gentzen
- Formalization of Mathematic (set theory, natural numbers, etc): Zermelo, Fraenkel, Peano

MATHEMATICAL LOGIC

The vocabulary of mathematics:

- In all isoscele triangles have two equal angles This triangle's angles are all different. Thus, it is not isoscele.
- If it is nice weather, I am definitely go climbing on the weekend.

Last weekend I didn't go climbing.

Thus, last weekend it wasn't nice weather.

P implies Q. Q does not hold, thus P also does not hold.

CONTENT AND FORM

'Inference' is not based on the truth of the content, but is only determined by the form.

- All students like holidays. There is a student that likes mathematic. Thus, there is a student who likes both holidays and math.
- Every polynomial equation has complex roots.
 This polynomial equation does not have real roots.
 Thus, all the roots of this equations are complex numbers.

The language of logic

- Logical operators: \land , \lor , \rightarrow , \neg
- ▶ Quantifiers: \forall , \exists
- ► Variables: *x*, *y*, ...
- ► Function symbols: *f*, *g*, ...
- Predicate symbols: P, Q, \ldots

EXAMPLE

Let the meaning of B(x, y) be 'x and y are siblings.'

Let the meaning of S(z) be that 'z is a student of JAIST'.

'Not all siblings of x are students of JAIST.'

 $\neg \forall y (B(x, y) \rightarrow S(y))$

DIFFERENCES IN THE MEANING

- ► $\forall x \exists y Q(x, y)$
- ► $\exists x \forall y Q(x, y)$

STUDENT'S LIFE

All students like holidays. There are students who like mathematics. Thus, there are students that like both holidays and mathematics.

S(x) Student, H(x) likes holidays, M(x) likes mathematics

 $[\forall x(S(x) \to H(x)) \land \exists x(S(x) \land M(x))] \to \exists x(S(x) \land H(x) \land M(x))$