

## L211 Logic and Mathematics

### 9. Lecture

Norbert PREINING

[preining@jaist.ac.jp](mailto:preining@jaist.ac.jp)

<http://www.preining.info/jaist/l211/2015e/>

## REPETITION

- ▶ injective
- ▶ surjective
- ▶ bijective
- ▶ comparing instead of counting

## DIAGONAL ARGUMENT: $\mathbb{N} \not\cong (0, 1)_{\mathbb{R}}$

- ▶ countable infinite set  $A$ : there exists a bijection  $f : \mathbb{N} \rightarrow A$   
Examples:  $\mathbb{N}, 2\mathbb{N}, \mathbb{P}, \mathbb{Q}$

- ▶ Assume:  $(0, 1)_{\mathbb{R}}$  is countable.  
Thus, there is a bijection  $f : \mathbb{N} \rightarrow (0, 1)_{\mathbb{R}}$

- ▶ Write all the reals in  $(0, 1)_{\mathbb{R}}$  in a table:

$n$	$f(n)$
1	0.31761342343...
2	0.11983947191...
3	0.01000000000...
$\vdots$	$\ddots$

## THE TABLE OF REALS

$n$	$f(n)$	$f(n) = e?$
1	0. $r_1^1 r_2^1 r_3^1 r_4^1 \dots$	$f(1) = e?$ no: $r_1^1 \neq s_1^1$
2	0. $r_1^2 r_2^2 r_3^2 r_4^2 \dots$	$f(2) = e?$ no: $r_1^2 \neq s_1^2$
3	0. $r_1^3 r_2^3 r_3^3 r_4^3 \dots$	$f(3) = e?$ no: $r_1^3 \neq s_1^3$
4	0. $r_1^4 r_2^4 r_3^4 r_4^4 \dots$	$f(4) = e?$ no: $r_1^4 \neq s_1^4$
$\vdots$	$\ddots$	$\vdots$
$n$	0. $r_1^n r_2^n r_3^n r_4^n \dots r_n^n \dots$	$f(n) = e?$ no: $r_n^n \neq s_n^n$

$d = 0.r_1^1 r_2^2 r_3^3 r_4^4 \dots r_n^n \dots$        $e = 0.s_1^1 s_2^2 s_3^3 s_4^4 \dots s_n^n \dots$

$r_1^1 \neq s_1^1, r_2^2 \neq s_2^2, \dots, r_n^n \neq s_n^n$

Conclusion:  $\forall n : f(n) \neq e$  thus,  $e \notin (0, 1)_{\mathbb{R}}$

## THE NUMBER OF NATURAL NUMBERS AND REALS

$\mathbb{N} \neq \mathbb{R}$

- ▶  $(0, 1) \cong \mathbb{R}$ :  $\tan \pi(x - 1/2)$
- ▶ diagonal method

$\mathbb{I} \cong \mathbb{R}$

- ▶ Because  $\mathbb{I}$  is infinite, there is a  $\mathbb{J}$  such that  $\mathbb{I} \cong \mathbb{J} + \mathbb{N}$   
( $A + B$  is a disjoint union)
- ▶ consequently,  $\mathbb{R} = \mathbb{I} + \mathbb{Q} \cong \mathbb{J} + \mathbb{N} + \mathbb{Q} \cong \mathbb{J} + \mathbb{N} \cong \mathbb{I}$

## POWERSET

- ▶ Powerset of  $A$ :  $\wp(A) = \{S : S \text{ is a subset of } A\}$
- ▶ if  $A \cong n$  holds, what is the number of elements in  $\wp(A)$ ?
- ▶ Are there sets of arbitrary cardinality?

For all  $A, A \prec \wp(A)$

thus

$\mathbb{N} \prec \wp(\mathbb{N}) \cong \mathbb{R} \prec \wp(\wp(\mathbb{N})) \prec \dots$

Is there a proof for  $\wp(\mathbb{N}) \cong \mathbb{R}$ ?

## CONTINUUM HYPOTHESIS

Is there a set that falls cardinality-wise between  $\mathbb{N}$  and  $\mathbb{R}$ ?

$$\exists A : |\mathbb{N}| \prec |A| \prec |\mathbb{R}|$$

Hilbert, Gödel, Cohen (1963)