L211 Logic and Mathematics

8. Lecture

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Last lectures

Numbers, number systems

Important points

- finite decimal, recurring decimals
- $\pi = 3.1415926535\ldots$
- Dedekind cut
- Base 10 number system
- Positional notation system
- Base 2, Base 16, Base 12, Base 60, Base 20, Base 27

Homework

Logic

From set theory to symbolic logic

Infinite sets

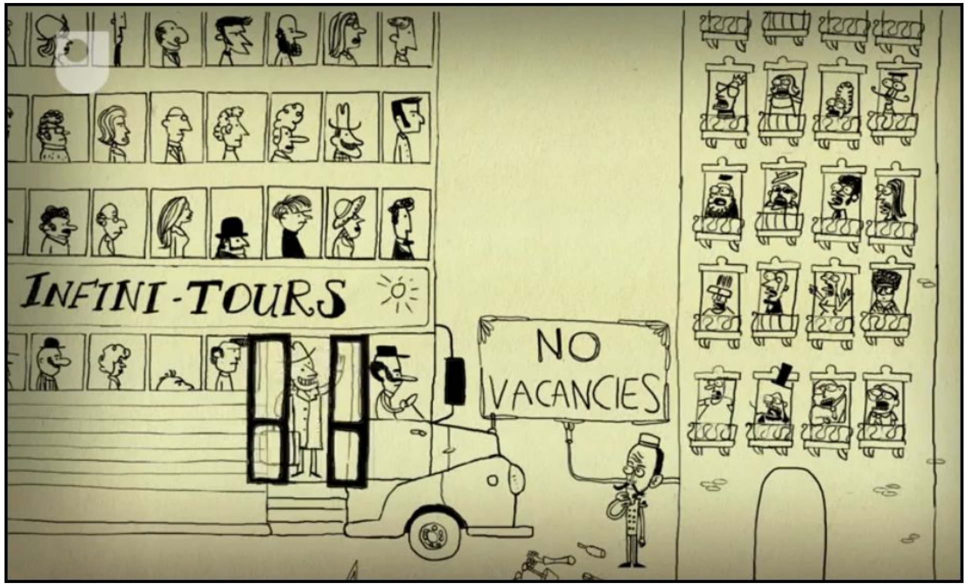
Number of elements

- Are there more elements in $\mathbb{Q}$ than in $\mathbb{N}$?
- Are there more elements in $\mathbb{R}$ than in $\mathbb{Q}$?

Possible answers

A1 all these sets are infinite, one cannot count them, so the above question does not make sense

A2 because $\mathbb{N} \subset \mathbb{Q} \subset \mathbb{R}$, the right side always has more elements than the left one



Hilbert's Infinite Hotel

The Hotel

A hotel with infinitely many rooms – for each natural number one

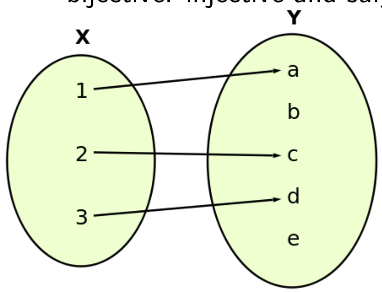
But all the rooms are occupied!

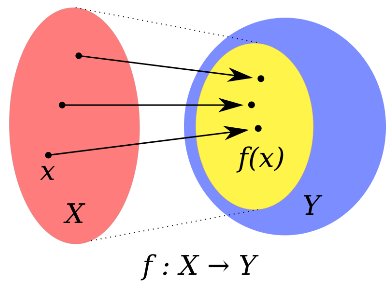
- k new guests want to stay over night – how should that be done?
- countable many new guests want to stay over night – how can we do that?

Injective and surjective

map:

- injective: a one to one mapping
- surjective: all elements are covered
- bijective: injective and surjective





Counting infinite sets

Equal cardinality

Take two sets A, B

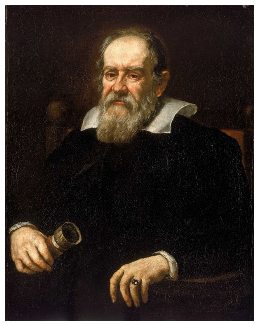
If there is a bijection $A \to B$, then A and B have equal cardinality: $A \cong B$

In case of finite sets, this means equal number of elements.

- $f : A \to B$ is an injection $\Leftrightarrow$
  if $x \neq y$ then $f(x) \neq f(y)$
- $f : A \to B$ is a surjection $\Leftrightarrow$
  for all $y \in B$ there is an $x \in A$ such that $y = f(x)$.

Examples for equal cardinality

- $(-1, 1)$ and $(0, 1)$ have equal cardinality
- Square numbers and natural numbers have equal cardinality – Galileo Galilei



Infinite and countable infinite

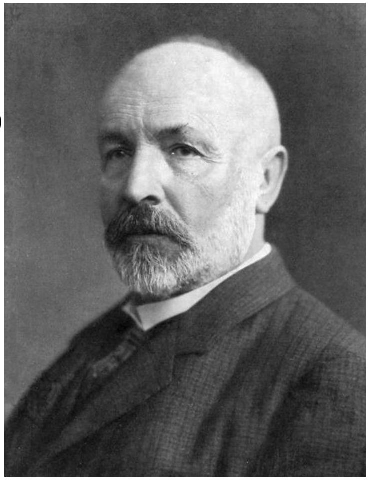
- $A \preceq B$: there is an injection $f : A \to B$
- Cantor-Bernstein theorem:
  $A \preceq B$ and $B \preceq A$, then $A \cong B$
- If $A \subseteq B$, then $A \preceq B$
- If $\mathbb{N} \preceq A$, then A is called infinite.
- If $\mathbb{N} \cong A$, then A is countable infinite.

Examples for countable infinite

- even numbers
- primes
- $\mathbb{N} \times \mathbb{N}$
- $\mathbb{Q}$

Georg Cantor

- *1845 St. Petersburg (Russia)
- †1918 Halle (Germany)
- German Mathematician
- Set theory
- Continuum hypothesis



The number of natural numbers and reals

$\mathbb{N} \neq \mathbb{R}$

- $(0, 1) \cong \mathbb{R}$: $\tan \pi(x - 1/2)$
- diagonal argument

$\mathbb{I} \cong \mathbb{R}$

- Since $\mathbb{I}$ is infinite, there is a set $\mathbb{J}$ such that $\mathbb{I} \cong \mathbb{J} + \mathbb{N}$
  ($A + B$ disjoint union)
- thus, $\mathbb{R} = \mathbb{I} + \mathbb{Q} \cong \mathbb{J} + \mathbb{N} + \mathbb{Q} \cong \mathbb{J} + \mathbb{N} \cong \mathbb{I}$