L211 Logic and Mathematics

5. Lecture

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Last lecture

Induction

Important points

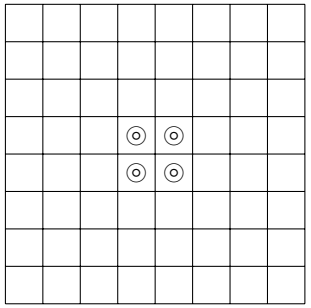
- write down that you use induction
- state the predicate/property you are going to prove
- prove $P(0)$
- prove $P(n) \to P(n+1)$
- qed $\square$
- the assumption needs to be used
- induction is not persuasion

Homework

Example

Stone Garden

Tiling a $2^n \times 2^n$ sized garden



Tile shape

All $2^n \times 2^n$ stone gardens can be tiled, so that in the center one piece remains.

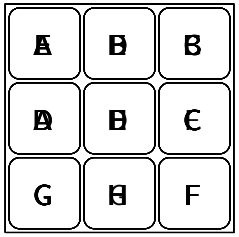
Conclusion

Sometimes it is better to use a stronger induction hypothesis!

Example

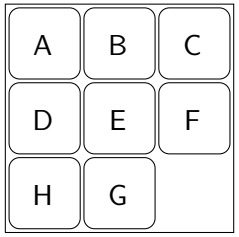
8-Puzzle

8-Puzzle

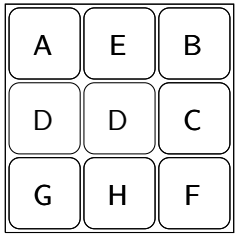


Finished!

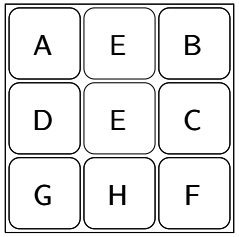
How about this one?



Row move

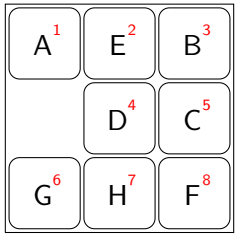


Column move

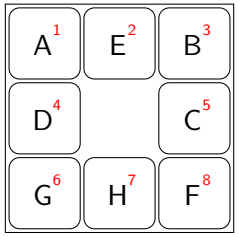


Where is the difference? the order changes!

Row move

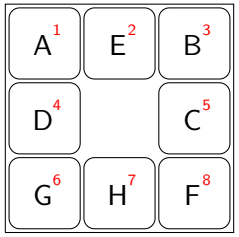


Row move

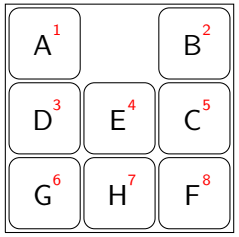


Lemma: During a row move, the order of all pairs remain the same.

Column move

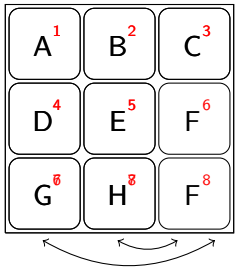


Column move



Lemma: During a column move, the ordering of two pairs changes.

Inverted pairs: How many inversions are there?



Theorem

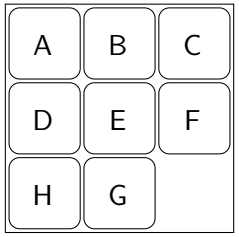
The parity of the number of order inversions does not change with any move.

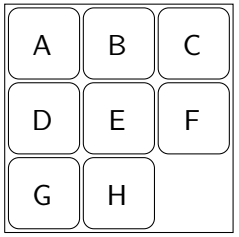
(If it is even at the beginning, it remains even.)

Proof:

- During a row move, the number of inversions does not change.
- During a column move, the number of inversions changes by $+2$, ± 0, or -2.

Applying the theorem





Number of order inversions: 1 Number of order inversions: 0

Thus, the left puzzle cannot be solved.

Conclusion

In case of recurrence/iterations, one needs to find a non-changing property and prove that it doesn't changes (parity in this case).

Induction principle

For a property $P(n)$ of natural numbers, if

$P(0)$ holds

and

for all natural numbers n, $P(n) \to P(n+1)$ holds

then

$P(n)$ holds for all natural numbers, that is $\forall n P(n)$ holds.

Example

Breaking down towers

Another method – strong induction

$P(0)$ holds

and

if for all k less than n, $P(k)$ holds, then also $P(n)$ holds

then

$P(n)$ holds for all natural numbers, that is $\forall n P(n)$ holds.