L211 Logic and Mathematics

3. Lecture

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Last lecture

From counting to the science of patterns

Important points

- fundamental theorem of arithmetic
- prime number theorem
  the frequency of prime number in the natural numbers is
  $$\pi(n) \sim \frac{n}{\ln n}$$
- Goldbach Conjecture

  Every even number bigger than 2 can be represented as the sum of two prime numbers.
- Completion:
  subtraction: $\mathbb{N} \to \mathbb{Z}$
  division: $\mathbb{Z} \to \mathbb{Q}$
  root: $\mathbb{Q} \to \mathbb{A}$, $\mathbb{A} \to \mathbb{R}$, $\mathbb{R} \to \mathbb{C}$

Proof

From the dictionary ...

Evidence or argument establishing or helping to establish a fact or the truth of a statement.

Convince others of the correctness.

All kinds of proof types

- Proof by trivialization: 'such a trivial thing does not need a proof'
- proof by usefulness: 'it would be so good if this is true'
- proof by necessity: 'if this is not true, that would be the end'
- proof by intimidation: 'don't be stupid, it's true, obviously!'
- proof by (wrong) analogy: 'that looks a bit like this one, thus it's true'
- proof by intuition: 'I have the feeling it's true'
- proof by authority: 'XXX has said it's true'

Last but not least ...

proof by love: 'I got a ring from my lover, so ...'

Mathematical proof

- confirm the correctness
- deduction/inference
  - from axioms
  - from already proven theorems or lemmas
  - each single step can be checked
- written so that it can be understood

Mathematical truth and falsity

TRUE

A proposition is either true or false. (No half-true!)

Connecting propositions

$\vee$	or	(disjunction)	$A \vee B$
$\wedge$	and	(conjunction)	$A \wedge B$
$\rightarrow$	implies	(implication)	$A \rightarrow B$
$\neg$	not	(negation)	$\neg A$

NECESSARY, SUFFICIENT, AND BOTH

necessary	A is a necessary condition for B $x^2 = 1$ is necessary for $x = 1$ $B \rightarrow A$
sufficient	A is a sufficient condition for B $x = 1$ is sufficient for $x^2 = 1$ $A \rightarrow B$
both	necessary and sufficient condition $A \leftrightarrow B$

TRUTH TABLES

	basic values
true	1
false	0

A	$\neg A$
0	1
1	0

Truth tables 2

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

A	B	$A \rightarrow B$
0	0	1
0	1	1
1	0	0
1	1	1

Truth table exercise

$$(P \vee Q) \rightarrow R$$

Truth tables 3

A	B	$A \leftrightarrow B$
0	0	1
0	1	0
1	0	0
1	1	1

Quantifiers

universal quantifier

symbol: $\forall$ 'for all'...
Example: $\forall n \in \mathbb{N} : P(x)$
For all natural number n, $P(n)$ (holds)

existential quantifier

symbol: $\exists$ 'exists'...
example: $\exists n \in \mathbb{N} : P(x)$
There is a natural number n such that $P(n)$ (holds).

Other operators: $\exists!$

Relation between quantifiers

negation of 'for all n, $P(n)$' ...

$$\neg \forall n P(n)$$

becomes 'there is an n such that $\neg P(n)$'

$$\neg \forall n P(n) \quad \Leftrightarrow \quad \exists n \neg P(n)$$

$$\neg \exists n P(n) \quad \Leftrightarrow \quad \forall n \neg P(n)$$

Proof methods

Direct proof

Theorem
If for two sets A and B we have that $A \subseteq B$ and $B \subseteq A$, then we also have $A = B$.

Proof

- (1) If $x \in A$, then due to $A \subseteq B$, also $x \in B$.
- By (1), all $x \in A$ are also $\in B$.
- (2) If $x \in B$, then due to $B \subseteq A$, also $x \in A$.
- By (2), all $x \in A$ are also $\in B$.
- By (1) and (2), we have $A = B$.

Direct proof

Theorem
$(A \subseteq B) \wedge (B \subseteq A) \Rightarrow (A = B)$

Proof

- (1) $x \in A \wedge A \subseteq B \Rightarrow x \in B$
- (2) $x \in B \wedge B \subseteq A \Rightarrow x \in A$
- (1) $\wedge$ (2) $\Rightarrow A = B$

Proof by contraposition

Direct proof

$$\text{Assumption} \implies \text{Conclusion}$$

Assuming the Assumption, we proof the Conclusion.

Basic pattern of the contraposition

$$A \to B \qquad \Leftrightarrow \qquad \neg B \to \neg A$$

Assuming the negation of the conclusion, we prove the negation of the assumption.

Confirmation of the contraposition

A	B	$A \to B$	$\neg A$	$\neg B$	$\neg B \to \neg A$
0	0	1	1	1	1
0	1	1	1	0	1
1	0	0	0	1	0
1	1	1	0	0	1

Example of contraposition

Theorem (n natural number)

If n^2 is even, then also n is even.

formula of the contraposition

If n is not even, then n^2 is also not even.

Proof

$$\begin{aligned} n \text{ is not even } n &= 2k+1 \text{ is assumed} \\ n^2 &= (2k+1)^2 \\ n^2 &= 4k^2+4k+1 \\ n^2 &= 2(2k^2+2k)+1 \\ &\text{thus } n^2 \text{ is not even, too.} \end{aligned}$$

Proof by contradiction

basic pattern

If we can deduce a contradiction assuming A , then A is false.

Example

$\sqrt{2}$ is an irrational number.

Proof

Assume $\sqrt{2}$ is a rational number . . .

Homework

Infinity of primes
The are infinitely many prime numbers.

Proof
Assume there are only finitely many primes, then ...

Summary

- Concept of proof in the society and in science is different
- Proofs go from axioms step by step
- there are various different proof methods

Exercise sheet

Wednesday lecture: Induction

Sources

- Ring: `http://www.fotocommunity.de/pc/pc/display/29133067`