DISSERTATION

Complete Recursive Axiomatizability of Gödel Logics

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Kurzfassung der Dissertation

Diese Dissertation beinhaltet eine vollständige Charakterisierung der rekursiv axiomatisierbaren Gödellogiken. Diese Logiken sind eine natürliche Klasse von mehrwertigen Logiken mit Wahrheitswerten aus $[0,1]$, die in vielen logischen und informatischen Zusammenhängen auftreten, z.B. als unmittelbare Erweiterungen der intuitionistischen Logik, als eine der grundlegenden Fuzzylogiken, sowie im Zusammenhang mit temporallogischen Fragestellungen, die auch für die automatische Verifikation von Programmen von Bedeutung sind. Sowohl die Existenz als auch die Nichtexistenz einer rekursiven Axiomatisierung für alle propositionalen Gödellogiken und Gödellogiken erster Ordnung werden beschrieben. Hierzu werden die topologischen Eigenschaften der zugrundeliegenden Wahrheitswertmengen untersucht und unter Zuhilfenahme von Konzepten der deskriptiven Mengenlehre sowie der Cantor-Bendixon Ableitung charakterisiert. Weiters wird die Kompaktheit der Folgerungsrelation von propositionalen Gödellogiken und Gödellogiken erster Ordnung charakterisiert.

Norbert Preining

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Dank ergeht in erster Linie an Bernhard Bodenstorfer für die detaillierte Durchsicht.

So long, and thanks for all the fish.

Contents

CHAPTER 1

Introduction and preliminaries

Gödel logics are one of the oldest and most interesting families of many-valued logics. Propositional finite-valued Gödel logics were introduced by Gödel in [Göd33] to show that intuitionistic logic does not have a characteristic finite matrix. They provide the first examples of intermediate logics (intermediate, that is, in strength between classical and intuitionistic logics). Dummett [Dum59] was the first to study infinite valued Gödel logics, axiomatizing the set of tautologies over infinite truth-value sets by intuitionistic logic extended by the linearity axiom $(A \supset B) \vee (B \supset A)$. Hence, infinite-valued propositional Gödel logic is also called Gödel-Dummett logic or Dummett's **LC**. In terms of Kripke semantics, the characteristic linearity axiom picks out those accessibility relations which are linear orders.

Quantified propositional Gödel logics and first-order Gödel logics are natural extensions of the propositional logics introduced by Gödel and Dummett. For both propositional quantified and first-order Gödel logics it turns out to be inevitable to consider more complex truth value sets than the standard unit interval.

Gödel logics occur in a number of different areas of logic and computer science. For instance, Dunn and Meyer [DM71] pointed out their relation to relevance logics; Visser [Vis82] employed **LC** in investigations of the provability logic of Heyting arithmetic; three-valued Gödel logic $\mathbf{G}_3$ has been used to model strong equivalence between logic programs. Furthermore, these logics have recently received increasing attention, both in terms of foundational investigations and in terms of applications, as they have been recognized as one of the most important formalizations of fuzzy logic [Háj98].

Perhaps the most surprising fact is that whereas there is only one infinite-valued propositional Gödel logic, there are infinitely many different logics at the first-order level [BLZ96b, BV99]. In the light of the general result of Scarpellini [Sca62] on non-axiomatizability, it is interesting that some of the infinite-valued Gödel logics belong to the limited class of recursively enumerable linearly ordered first-order logics [Hor69, TT84].

In this thesis we present a complete characterization of Gödel logics with respect to complete recursive axiomatizability. We will describe those logics which admit a complete recursive axiomatization[1] in terms of topological structures of the underlying truth value sets. Furthermore we analyze the entailment relation of propositional and first-order Gödel logics and prove that only those which admit a complete recursive axiomatization admit compact entailment relations.

1.1 Overview on the results

A fundamental question for any logic is whether a formalization with axioms and rules is complete, i.e. if all valid sentences in this logic can be derived from the given axioms. The results in this thesis characterize in detail all the Gödel logics – propositional, first-order, entailment – which can be axiomatized.

Basis of our work is the result obtained by Takano [Tak87] for the most important Gödel logic over the real interval $[0, 1]$.

In the present chapter we introduce the syntax of Gödel logics. Extensions using propositional quantifiers and the Δ operator are considered. Furthermore, we discuss the relationship between different Gödel logics. After stating axioms and deduction systems we present the above mentioned completeness proof of Takano in Section 1.5 and extend it to the case of Gödel logics with Δ.

In Chapter 2, notions of topology and order theory are introduced which are essential for the content of the subsequent chapters. The relationship between dense linear orderings and perfect sets is clarified.

Chapter 3 deals with propositional Gödel logics, presents a variant of the proof of completeness, and discusses in short quantified propositional Gödel logics.

Chapter 4 deals with propositional entailments.

Chapter 5 contains the main results for first-order Gödel logics. In Section 5.3 the impossibility to give a complete recursive axiomatiza-

[1] We will call these logics *complete*.

tion of Gödel logics with countable truth value sets is derived. For uncountable truth value sets the topological structure of the truth value set provides a criterion to distinguish axiomatizable logics. Completeness proofs for the various different logics and an extension with Δ completes this chapter.

In Chapter 6 the entailment relation of first-order Gödel logics are discussed and compactness is related to complete recursive axiomatizability.

Finally an overview of the achieved results is presented in the table on p. 60.

1.2 Definition of Gödel logics

1.2.1 The Gödel implication

Many-valued logics differ mainly in their definition of the truth function for implication. The truth function for Gödel implication is of particular interest as it can be 'deduced' from simple properties of the evaluation and the entailment relation (for details on entailment see Chapter 4), which has been observed by G. Takeuti.

LEMMA 1.1 *The Gödel implication given by*

$$\mathcal{I}(A \supset B) = \begin{cases} \mathcal{I}(B) & \text{if } \mathcal{I}(A) > \mathcal{I}(B) \\ 1 & \text{if } \mathcal{I}(A) \leq \mathcal{I}(B) \end{cases}$$

is the only definition of the truth function for implication which gives an entailment relation with the following conditions:

1. *The interpretation of the implication is* 1 *iff the implication of the antecedent is less or equal to the interpretation of the succedent, i.e.* $\mathcal{I}(A) \leq \mathcal{I}(B) \Leftrightarrow \mathcal{I}(A \supset B) = 1$
2. $\Pi \cup \{A\} \Vdash B \Leftrightarrow \Pi \Vdash A \supset B$
3. $\Pi \Vdash B \Rightarrow \min\{\mathcal{I}(A) : A \in \Pi\} \leq \mathcal{I}(B)$
   (and if $\Pi = \emptyset \Rightarrow 1 \leq \mathcal{I}(B)$*, this is in fact the definition of entailment).*

PROOF: From $B \Vdash B$ we obtain $B, A \Vdash B$ and with property 2, $B \Vdash A \supset B$, therefore,

$$\mathcal{I}(B) \leq \mathcal{I}(A \supset B).$$

From $A \supset B \Vdash A \supset B$ with property 2 again we obtain $A \supset B, A \Vdash B$ and from property 3

$$\min\{\mathcal{I}(A \supset B), \mathcal{I}(A)\} \leq \mathcal{I}(B),$$

which together with property 1 gives the definition of the Gödel implication. □

In other words, if we want a deduction system which extends the classical implication in the sense that if the antecedent is less true than the succedent, the whole formula is true (1), which has a deduction theorem (2) and which has an entailment property, then we obtain Gödel implication. This shows us that Gödel logics are in fact very general. All other many-valued logics fail in one of the above points.

1.2.2 Syntax and semantics for propositional Gödel logics

The language for propositional Gödel logics is a standard propositional language:

DEFINITION 1.2 *The language L^0 for propositional Gödel logics consists of the constant $\bot$, countably many propositional variables ($p_1, p_2, \dots$) and the connectives $\wedge$, $\vee$ and $\supset$. The set of well formed formulas, denoted by* $\mathrm{Frm}(L^0)$*, are defined as usual for a propositional logic.*

NOTE: As usual in the area of intuitionistic logic we do *not* have the negation in the language, but it is an abbreviation for

$$\neg p \leftrightarrow p \supset \bot.$$

Furthermore we will use $p \prec q$ as an abbreviation for

$$p \prec q \leftrightarrow (q \supset p) \supset q$$

and $\top \leftrightarrow p \supset p$.

DEFINITION 1.3 *Let $V \subseteq [0,1]$ be some set of truth values which contains 0 and 1. A propositional Gödel valuation $\mathcal{I}_V^0$ (short valuation) based on V is a function from the set of propositional variables into V with $\mathcal{I}_V^0(\bot) = 0$. This valuation can be extended to a function mapping formulas from* $\mathrm{Frm}(L^0)$ *into V as follows:*

$$\begin{aligned}
\mathcal{I}_V^0(A \wedge B) &= \min\{\mathcal{I}_V^0(A), \mathcal{I}_V^0(B)\} \\
\mathcal{I}_V^0(A \vee B) &= \max\{\mathcal{I}_V^0(A), \mathcal{I}_V^0(B)\} \\
\mathcal{I}_V^0(A \supset B) &= \begin{cases} \mathcal{I}_V^0(B) & \text{if } \mathcal{I}_V^0(A) > \mathcal{I}_V^0(B) \\ 1 & \text{if } \mathcal{I}_V^0(A) \le \mathcal{I}_V^0(B). \end{cases}
\end{aligned}$$

A formula is called valid with respect to V if it is mapped to 1 for all valuations based on V.

The set of all formulas which are valid with respect to V will be called the propositional Gödel logic based on V and will be denoted by $\mathbf{G}^0_V$.

The validity of a formula A with respect to V will be denoted by

$$\vDash^0_V A \quad \textit{or} \quad \vDash_{\mathbf{G}^0_V} A.$$

NOTE: The extension of the valuation $\mathcal{I}^0_V$ to formulas provides the following truth functions:

$$\mathcal{I}^0_V(\neg A) = \begin{cases} 0 & \text{if } \mathcal{I}^0_V(A) > 0 \\ 1 & \text{otherwise} \end{cases}$$

$$\mathcal{I}^0_V(A \prec B) = \begin{cases} 1 & \text{if } \mathcal{I}^0_V(A) < \mathcal{I}^0_V(B) \text{ or } \mathcal{I}^0_V(A) = \mathcal{I}^0_V(B) = 1 \\ \mathcal{I}(B) & \text{otherwise} \end{cases}$$

Thus, the intuition behind $A \prec B$ is that A is strictly less than B, or both are equal to 1.

1.2.3 Syntax and semantics for quantified propositional Gödel logics

In *classical* propositional logic one defines $\exists p A(p)$ by $A(\bot) \vee A(\top)$ and $\forall p A(p)$ by $A(\bot) \wedge A(\top)$. In other words, propositional quantification is semantically defined by the supremum and infimum, respectively, of truth functions (with respect to the usual ordering $0 < 1$ over the classical truth-values $\{0, 1\}$). This can be extended to Gödel logic by using fuzzy quantifiers. Syntactically, this means that we allow formulas $\forall p A$ and $\exists p A$ in the language. Free and bound occurrences of variables are defined in the usual way. So the language for propositional *quantified* Gödel logics consists of the following extensions to the language L^0:

DEFINITION 1.4 *The language L^{qp} for quantified propositional Gödel logics contains the language L^0 and the quantifiers $\forall$ and $\exists$. The set of well formed formulas, denoted by $\mathrm{Frm}(L^{qp})$, is the minimal set fulfilling the following conditions: All propositional variables and constants are well formed formulas, and if A and B are well formed, p a propositional variable, then also $A \wedge B$, $A \vee B$, $A \supset B$, and $\exists p A$ and $\forall p A$ are well formed formulas.*

The semantics of propositional quantifiers is defined analogously to that of first-order quantifiers as the infimum and supremum of the corresponding distribution. In this context a distribution of a formula A and free propositional variable p with respect to an interpretation $\mathcal{I}$ is defined as

$$\mathrm{Distr}_{\mathcal{I}}(A(p)) = \{\mathcal{I}'(A(p)) : \mathcal{I}' \sim_p \mathcal{I}\}$$

where $\mathcal{I}' \sim_p \mathcal{I}$ means that $\mathcal{I}'$ is exactly as $\mathcal{I}$ with the possible exception of the truth-value assigned to p.

Extending the definition of valuation to the quantified propositional case we have to take care that the truth value set has to be closed under infima and suprema, otherwise the valuation of quantified formulas will not necessarily be well defined.

DEFINITION 1.5 *Let $V \subseteq [0,1]$ be some set of truth values which contains 0 and 1 and is closed under infima and suprema. A valuation $\mathcal{I}_V^{qp}$ based on V is a function from the set of propositional variables into V with $\mathcal{I}_V^{qp}(\bot) = 0$. This valuation can be extended to a function mapping formulas from* $\mathrm{Frm}(L^{qp})$ *into V as follows:*

$$\begin{aligned}
\mathcal{I}_V^{qp}(A \wedge B) &= \min\{\mathcal{I}_V^{qp}(A), \mathcal{I}_V^{qp}(B)\} \\
\mathcal{I}_V^{qp}(A \vee B) &= \max\{\mathcal{I}_V^{qp}(A), \mathcal{I}_V^{qp}(B)\} \\
\mathcal{I}_V^{qp}(A \supset B) &= \begin{cases} \mathcal{I}_V^{qp}(B) & \textit{if } \mathcal{I}_V^{qp}(A) > \mathcal{I}_V^{qp}(B) \\ 1 & \textit{if } \mathcal{I}_V^{qp}(A) \leq \mathcal{I}_V^{qp}(B) \end{cases} \\
\mathcal{I}_V^{qp}(\forall p A(p)) &= \inf \mathrm{Distr}_{\mathcal{I}_V^{qp}}(A(p)) \\
\mathcal{I}_V^{qp}(\exists p A(p)) &= \sup \mathrm{Distr}_{\mathcal{I}_V^{qp}}(A(p))
\end{aligned}$$

A formula is called valid with respect to V if it is mapped to 1 for all valuations based on V.

The set of all formulas which are valid with respect to V will be called the propositional quantified Gödel logic based on V and will be denoted by $\mathbf{G}_V^{qp}$.

The validity of a propositional quantified formula A with respect to V will be denoted by

$$\models_V^{qp} A \quad \textit{or} \quad \models_{\mathbf{G}_V^{qp}} A.$$

1.2.4 Syntax and semantics for first-order Gödel logics

DEFINITION 1.6 *The language L^{fo} for first-order Gödel logics contains countably many free variables (a, b, ...), countably many bound variables (x, y, ...), countably many function symbols (f, g, ...), predicate symbols (P, Q, R, ...), the 0-placed predicate constant $\bot$, the connectives $\wedge$, $\vee$, $\supset$ and the quantifiers $\forall$ and $\exists$. Terms, atomic formulas, formulas are defined as usual.*

Extending the definition of valuation to the first-order case we have again to take care that the truth value set has to be closed under infima and suprema, otherwise the valuation of quantified formulas will not necessarily be well defined.

Definition 1.7 *Let* $V \subseteq [0,1]$ *be some set of truth values which contains* 0 *and* 1 *and is closed under infima and suprema. A* first-order interpretation $\mathcal{I}_V^{fo} = \langle D, \mathbf{s} \rangle$ based on V *is given by the* domain D *and the* valuation function $\mathbf{s}$ *which maps* n*-ary relation symbols to functions* $D^n \to V$, $\mathbf{s}(\bot) = 0$, n*-ary function symbols to functions from* D^n *to* D, *and constants of* L^1 *and variables to elements of* D. L^1 *is* L *extended by constant symbols for all* $d \in D$*; if* $d \in D$*, then* $\mathbf{s}(d) = d$.

$\mathbf{s}$ *can be extended in the obvious way to a function on all terms. The valuation for formulas is defined as follows:*

$$\begin{aligned}
\mathcal{I}_V^{fo}(P(t_1, \dots, t_n)) &= \mathbf{s}(P)(\mathbf{s}(t_1), \dots, \mathbf{s}(t_n)) \\
\mathcal{I}_V^{fo}(A \wedge B) &= \min\{\mathcal{I}_V^{fo}(A), \mathcal{I}_V^{fo}(B)\} \\
\mathcal{I}_V^{fo}(A \vee B) &= \max\{\mathcal{I}_V^{fo}(A), \mathcal{I}_V^{fo}(B)\} \\
\mathcal{I}_V^{fo}(A \supset B) &= \begin{cases} \mathcal{I}_V^{fo}(B) & \text{if } \mathcal{I}_V^{fo}(A) > \mathcal{I}_V^{fo}(B) \\ 1 & \text{if } \mathcal{I}_V^{fo}(A) \leq \mathcal{I}_V^{fo}(B) \end{cases} \\
\mathcal{I}_V^{fo}(\forall x A(x)) &= \inf \mathrm{Distr}_{\mathcal{I}_V^{fo}}(A(a)) \\
\mathcal{I}_V^{fo}(\exists x A(x)) &= \sup \mathrm{Distr}_{\mathcal{I}_V^{fo}}(A(a))
\end{aligned}$$

where the set $\mathrm{Distr}_{\mathcal{I}_V^{fo}}(A(a)) = \{\mathcal{I}_V^{fo}(A(d)) : d \in D\}$ *is called* distribution *of* $A(x)$.

A formula is called valid with respect to V *if it is mapped to* 1 *for all valuations based on* V.

The set of all formulas which are valid with respect to V *will be called the first-order Gödel logic based on* V *and will be denoted by* $\mathbf{G}_V^{fo}$.

The validity of a first-order formula A *with respect to* V *will be denoted by*

$$\vDash_V^{fo} A \quad \text{or} \quad \vDash_{\mathbf{G}_V^{fo}} A.$$

Note: We will not write the superscripts $0, qp, fo$ and the subscript V if it is obvious from the context (which will generally be the case).

1.2.5 Extension with Δ

In [TT86, Baa96, Tit97] the Δ-operator has been introduced to the language of logic expressing that the truth value of the operand is 1. I.e. a new unary operator is introduced

$$\Delta A$$

and the interpretation is extended as following:

$$\mathcal{I}(\Delta A) = \begin{cases} 1 & \text{if } \mathcal{I}(A) = 1 \\ 0 & \text{otherwise} \end{cases}$$

We will denote logics containing Δ by a superscribed Δ, e.g. the logic $\mathbf{G}_V$ with Δ will be denoted by $\mathbf{G}_V^\Delta$. It has been introduced for symmetry reasons, since in (plain) Gödel logic the truth value 0 can be distinguished from other values, but not the truth value 1, this can only be done using the Δ-operator.

LEMMA 1.8 *The operator Δ is not expressible in the language of Gödel logics.*

PROOF: All truth functions in one variable are given by $f(x) = x$, $f(x) = \neg x$, $f(x) = \neg\neg x$, $f(x) = 0$ or $f(x) = 1$, but none of these truth functions coincide with the truth function for Δ, therefore, the Δ operator is not expressible. □

1.3 Relationships between Gödel logics

As we will see in Chapter 3, the relationships between finite and infinite valued *propositional* Gödel logics are well understood. Any choice of an infinite set of truth-values results in the same propositional Gödel logic, viz., Dummett's **LC**. **LC** was defined using the set of truth-values $V_\downarrow$ (see below). Furthermore, we know that **LC** is the intersection of all finite-valued propositional Gödel logics, and that it is axiomatized by intuitionistic propositional logic **IPL** plus the schema $(A \supset B) \vee (B \supset A)$. **IPL** is contained in all Gödel logics.

In the first-order case, the relationships are somewhat more interesting. First of all, let us note the following fact corresponding to the end of the previous paragraph:

PROPOSITION 1.9 *Intuitionistic predicate logic* **IL** *is contained in all first-order Gödel logics.*

PROOF: The axioms and rules of **IL** are sound for the Gödel truth functions. □

As a consequence of this proposition, we will be able to use any intuitionistically sound rule and intuitionistically true formula when working in any of the Gödel logics.

We can consider special truth value sets which will act as prototypes for other logics. This is due to the fact that the logic is defined extensionally as the set of formulas valid in this truth value set, so the Gödel logics on different truth value sets may coincide.

$$
\begin{aligned}
V_{\mathbb{R}} &= [0,1] \\
V_{\downarrow} &= \{1/k : k \geq 1\} \cup \{0\} \\
V_{\uparrow} &= \{1 - 1/k : k \geq 1\} \cup \{1\} \\
V_m &= \{1 - 1/k : 1 \leq k \leq m-1\} \cup \{1\}
\end{aligned}
$$

The corresponding Gödel logics are $\mathbf{G}_{\mathbb{R}}$, $\mathbf{G}_{\downarrow}$, $\mathbf{G}_{\uparrow}$, $\mathbf{G}_m$. $\mathbf{G}_{\mathbb{R}}$ is the *standard* Gödel logic, $\mathbf{G}_{\downarrow}$ is the logic of well-founded linearly ordered Kripke-semantics with constant domains (for Kripke models see [Min00]). As this logic, the logic of well-founded linearly ordered Kripke-semantics is better known in the propositional case as Dummett's **LC** [Dum59], we want to exhibit this equivalence:

LEMMA 1.10 *The intuitionistic logic of well-founded linearly ordered Kripke-semantics with constant domains is equivalent to* $\mathbf{G}_{\downarrow}$.

PROOF: Let (W, R, U) be a Kripke model where the accessibility relation R is linearly ordered, i.e. for all $w, w' \in W$ either $R(w, w')$ or $R(w', w)$. We will call the first ω worlds 1, 2, ..., i.e. $W = \{1, 2, 3 \ldots\}$. We have to show that if a formula is not valid in one of these linearly ordered Kripke models with constant domains, then it is not valid in $\mathbf{G}_{\downarrow}$, and vice versa.

Assume that A is not valid in a Kripke model (W, R, U) as given above and assume that $\mathcal{I}_K(A, w)$ gives the truth-value of A in world w in this model (0 or 1). Define a function $\varphi : W \to V_{\downarrow}$, $\varphi(w) = 1/w$ and the interpretation of a formula A in the Gödel logic $\mathbf{G}_{\downarrow}$ as the image under φ of the smallest world such that A is true in this world, or 0 if no such world exists, i.e.

$$
\mathcal{I}_G(A) = \begin{cases} \varphi(\mu_w(\mathcal{I}_K(A, w) = 1)) & \text{if such a world exists} \\ 0 & \text{otherwise} \end{cases}
$$

where μ is the minimization operator. It is obvious that if A is not valid in the Kripke model, i.e. it is not true in the first world 1, then its truth value in $\mathbf{G}_{\downarrow}$ will also be less than 1. Thus, we only have to show that the function defined above in fact is an interpretation:

$$
\begin{aligned}
\mathcal{I}_G(A \wedge B) &= \varphi(\mu_w(\mathcal{I}_K(A \wedge B, w) = 1)) \\
&= \varphi(\max\{\mu_w(\mathcal{I}_K(A, w) = 1), \mu_w(\mathcal{I}_K(B, w) = 1)\}) \\
&= \min\{\varphi(\mu_w(\mathcal{I}_K(A, w) = 1)), \varphi(\mu_w(\mathcal{I}_K(B, w) = 1))\} \\
&= \min\{\mathcal{I}_G(A), \mathcal{I}_G(B)\}
\end{aligned}
$$

The change from max to min is due to the function φ which inverts the argument. For $\mathcal{I}_G(A \vee B)$ the computation is analogous. For $\perp$ remember that $\perp$ is evaluated to false in all worlds of a Kripke model, thus the Gödel interpretation is 0.

For the implication and the quantifiers it has to be noted that their interpretation has to be computed globally, i.e. the interpretation of an implication is valid in a world w if and only if the *local* interpretation (denoted by $\mathcal{I}_K^*$) is valid in w and all worlds which are reachable, i.e. in the case of linearly ordered Kripke structures, in all following worlds.

Consider now the implication $A \supset B$: Let $w_A = \mu_w(\mathcal{I}_K(A) = 1)$ and $w_B = \mu_w(\mathcal{I}_K(B) = 1)$ the worlds where A and B, respectively, turns true. If $w_B \le w_A$, i.e. $R(w_B, w_A)$ holds, then for all worlds w' the *local* evaluation $\mathcal{I}_K^*(A \supset B, w')$ is true, because B is true before A becomes true. Thus $\mathcal{I}_K(A \supset B, 1)$ is true and $\mathcal{I}_G(A \supset B) = \varphi(1) = 1$. On the other hand, if $w_B > w_A$, then the local interpretation $\mathcal{I}_K^*$ of $A \supset B$ is true before w_A and after w_B, thus the *global* interpretation $\mathcal{I}_K$ of $A \supset B$ will only be true in and after w_B, thus the Gödel interpretation $\mathcal{I}_G(A \supset B) = \varphi(w_B) = \mathcal{I}_G(B)$.

For the universal quantifier consider

$$\begin{aligned}\mathcal{I}_G(\forall x A(x)) &= \varphi(\mu_w(\mathcal{I}_K(\forall x A(x), w) = 1)) \\ &= \varphi(\mu_w(\forall l > w \forall c \in U \mathcal{I}_K(A(c), l) = 1))\end{aligned}$$

due to the constant domain property it is enough to consider only the world w

$$\begin{aligned}&= \varphi(\mu_w(\forall c \in U \mathcal{I}_K(A(c), w) = 1)) \\ &= \varphi(\sup_{c\in U} \mu_w(\mathcal{I}_K(A(c), w) = 1)) \\ &= \inf_{c\in U} \varphi(\mu_w(\mathcal{I}_K(A(c), w) = 1)) \\ &= \inf_{c\in U} \mathcal{I}_G(A(c))\end{aligned}$$

and for the existential an analogous computation. Here the requirement of constant domains is of importance. Thus, we have shown that if a linearly ordered Kripke structure with constant domains is a counter-model to a formula A, then we can give a counter-model in $\mathbf{G}_\downarrow$.

For the reverse direction we assume that $\mathcal{I}_G$ is a counter-model in $\mathbf{G}_\downarrow$ and we will give a Kripke model which is also a counter model. The universe for all worlds is the universe of $\mathbf{G}_\downarrow$, the worlds are $W = \{1, 2, 3, \ldots\}$, the accessibility relation is the successor, and the interpretation is defined as

$$\mathcal{I}_K(A, w) = \begin{cases} 0 & w < 1/\mathcal{I}_G(A) \\ 1 & w \ge 1/\mathcal{I}_G(A) \end{cases}.$$

It is easy to verify that this in fact gives a interpretation in the Kripke structure, completing the proof. □

The logic $\mathbf{G}_\downarrow$ also turns out to be closely related to some temporal logics [BLZ96b, BLZ96a]. $\mathbf{G}_\uparrow$ is the intersection of all finite-valued first-order Gödel logics as shown in Theorem 1.13.

PROPOSITION 1.11 *The following strict containment relationships hold:*

1. $\mathbf{G}_m \supsetneq \mathbf{G}_{m+1}$,
2. $\mathbf{G}_m \supsetneq \mathbf{G}_\uparrow \supsetneq \mathbf{G}_\mathbb{R}$,
3. $\mathbf{G}_m \supsetneq \mathbf{G}_\downarrow \supsetneq \mathbf{G}_\mathbb{R}$.

PROOF: The only non-trivial part is proving that the containments are strict. For this note that

$$(A_1 \supset A_2) \vee \ldots \vee (A_m \supset A_{m+1})$$

is valid in $\mathbf{G}_m$ but not in $\mathbf{G}_{m+1}$. Furthermore, let

$$\begin{aligned} C_\uparrow &= \exists x(A(x) \supset \forall y\, A(y))\text{and} \\ C_\downarrow &= \exists x(\exists y\, A(y) \supset A(x)). \end{aligned}$$

$C_\downarrow$ is valid in all $\mathbf{G}_m$ and in $\mathbf{G}_\uparrow$ and $\mathbf{G}_\downarrow$; $C_\uparrow$ is valid in all $\mathbf{G}_m$ and in $\mathbf{G}_\uparrow$, but not in $\mathbf{G}_\downarrow$; neither is valid in $\mathbf{G}_\mathbb{R}$ ([BLZ96b], Corollary 2.9). □

The formulas $C_\uparrow$ and $C_\downarrow$ are of some importance in the study of first-order infinite-valued Gödel logics. $C_\uparrow$ expresses the fact that every infimum in the set of truth values is a minimum, and $C_\downarrow$ states that every supremum (except possibly 1) is a maximum. The intuitionistically admissible quantifier shifting rules are given by the following implications and equivalences:

$$\begin{array}{rcl} (\forall x\, A(x) \wedge B) & \equiv & \forall x(A(x) \wedge B) \\ (\exists x\, A(x) \wedge B) & \equiv & \exists x(A(x) \wedge B) \\ (\forall x\, A(x) \vee B) & \supset & \forall x(A(x) \vee B) \\ (\exists x\, A(x) \vee B) & \equiv & \exists x(A(x) \vee B) \\ (B \supset \forall x\, A(x)) & \equiv & \forall x(B \supset A(x)) \\ (B \supset \exists x\, A(x)) & \subset & \exists x(B \supset A(x)) \\ (\forall x\, A(x) \supset B) & \subset & \exists x(A(x) \supset B) \\ (\exists x\, A(x) \supset B) & \equiv & \forall x(A(x) \supset B) \end{array}$$

The remaining three are:

$$\begin{array}{rclr} (\forall x\, A(x) \vee B) & \subset & \forall x(A(x) \vee B) & (S_1) \\ (B \supset \exists x\, A(x)) & \supset & \exists x(B \supset A(x)) & (S_2) \\ (\forall x\, A(x) \supset B) & \supset & \exists x(A(x) \supset B) & (S_3) \end{array}$$

Of these, S_1 is valid in any Gödel logic. S_2 and S_3 imply and are implied by $C_\downarrow$ and $C_\uparrow$, respectively (take $\exists y\, A(y)$ and $\forall y\, A(y)$, respectively, for B). S_2 and S_3 are, respectively, both valid in $\mathbf{G}_\uparrow$, invalid and valid in $\mathbf{G}_\downarrow$, and both invalid in $\mathbf{G}_\mathbb{R}$. Thus we obtain

COROLLARY 1.12 $\mathbf{G}_\uparrow$ *is the only Gödel logic where every formula is equivalent to a prenex formula with the same propositional matrix.*

We now also know that $\mathbf{G}_\uparrow \neq \mathbf{G}_\downarrow$. In fact, we have $\mathbf{G}_\downarrow \subsetneq \mathbf{G}_\uparrow$; this follows from the following theorem.

THEOREM 1.13

$$\mathbf{G}_\uparrow = \bigcap_{m\geq 2} \mathbf{G}_m$$

PROOF: By Proposition 1.11, $\mathbf{G}_\uparrow \subseteq \bigcap_{m\geq 2} \mathbf{G}_m$. We now prove the reverse inclusion. Assume that there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \not\models A$, we want to give an interpretation $\mathcal{I}'$ such that $\mathcal{I}' \not\models A$ and $\mathcal{I}'$ is a $\mathbf{G}_m$ interpretation for some m.

LEMMA 1.14 *If all infima in the truth value set are minima or A contains no quantifiers, and A evaluates to some $v < 1$ in $\mathcal{I}$, then A also evaluates to v in $\mathcal{I}_v$ where*

$$\mathcal{I}_v(P) = \begin{cases} 1 & \text{if } \mathcal{I}(P) > v \\ \mathcal{I}(P) & \text{otherwise} \end{cases}$$

for P atomic sub-formula of A.

PROOF: We prove by induction on the complexity of formulas that any sub-formula F of A with $\mathcal{I}(F) \leq v$ has $\mathcal{I}'(F) = \mathcal{I}(F)$. This is clear for atomic sub-formulas. We distinguish cases according to the logical form of F:

$F \equiv D \wedge E$. If $\mathcal{I}(F) \leq v$, then, without loss of generality, assume $\mathcal{I}(F) = \mathcal{I}(D) \leq \mathcal{I}(E)$. By induction hypothesis, $\mathcal{I}'(D) = \mathcal{I}(D)$ and $\mathcal{I}'(E) \geq \mathcal{I}(E)$, so $\mathcal{I}'(F) = \mathcal{I}(F)$. If $\mathcal{I}(F) > v$, then $\mathcal{I}(D) > v$ and $\mathcal{I}(E) > v$, by induction hypothesis $\mathcal{I}'(D) = \mathcal{I}'(E) = 1$, thus, $\mathcal{I}'(F) = 1$.

$F \equiv D \vee E$. If $\mathcal{I}(F) \leq v$, then, without loss of generality, assume $\mathcal{I}(F) = \mathcal{I}(D) \geq \mathcal{I}(E)$. By induction hypothesis, $\mathcal{I}'(D) = \mathcal{I}(D)$ and $\mathcal{I}'(E) = \mathcal{I}(E)$, so $\mathcal{I}'(F) = \mathcal{I}(F)$. If $\mathcal{I}(F) > v$, then, again without loss of generality, $\mathcal{I}(F) = \mathcal{I}(D) > v$, by induction hypothesis $\mathcal{I}'(D) = 1$, thus, $\mathcal{I}'(F) = 1$.

$F \equiv D \supset E$. Since $v < 1$, we must have $\mathcal{I}(D) > \mathcal{I}(E) = \mathcal{I}(F)$. By induction hypothesis, $\mathcal{I}'(D) \geq \mathcal{I}(D)$ and $\mathcal{I}'(E) = \mathcal{I}(E)$, so $\mathcal{I}'(F) = \mathcal{I}(F)$. If $\mathcal{I}(F) > v$, then $\mathcal{I}(D) \geq \mathcal{I}(E) = \mathcal{I}(F) > v$, by induction hypothesis $\mathcal{I}'(D) = \mathcal{I}'(E) = \mathcal{I}'(F) = 1$.

$F \equiv \exists x D(x)$. First assume that $\mathcal{I}(F) \leq v$. Since $D(c)$ evaluates to a value less or equal to v in $\mathcal{I}$ and, by induction hypothesis, in $\mathcal{I}'$ also the supremum of these values is less or equal to v in $\mathcal{I}'$, thus $\mathcal{I}'(F) = \mathcal{I}(F)$. If $\mathcal{I}(F) > v$, then there is a c such that $\mathcal{I}(D(c)) > v$, by induction hypothesis $\mathcal{I}'(D(c)) = 1$, thus, $\mathcal{I}'(F) = 1$.

$F \equiv \forall x D(x)$. This is the crucial part. First assume that $\mathcal{I}(F) < v$. Then there is a witness c such that $\mathcal{I}(F) \leq \mathcal{I}(D(c)) < v$ and, by induction hypothesis, also $\mathcal{I}'(D(c)) < v$ and therefore, $\mathcal{I}'(F) = \mathcal{I}(F)$. For $\mathcal{I}(F) > v$ it is obvious that $\mathcal{I}'(F) = \mathcal{I}(F) = 1$. Finally assume that $\mathcal{I}(F) = v$. If this infimum would be proper, i.e. no minimum, then the value of all witnesses under $\mathcal{I}'$ would be 1, but the value of F under $\mathcal{I}'$ would be v, which would contradict the definition of the semantic of the $\forall$ quantifier. Since all infima are minima, there is a witness c such that $\mathcal{I}(D(c)) = v$ and therefore, also $\mathcal{I}'(D(c)) = v$ and thus $\mathcal{I}'(F) = \mathcal{I}(F)$. □

Now suppose there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \not\models A$, let $\mathcal{I}(A) = v$. Then the interpretation $\mathcal{I}'$ given in the above lemma also is a counterexample for A. Since there are only finitely many truth values below v in $V_\uparrow$, say $v = 1 - 1/k$, $\mathcal{I}'$ is a $\mathbf{G}_{k+1}$ interpretation with $\mathcal{I}' \not\models A$. This completes the proof of the theorem. □

COROLLARY 1.15 $\mathbf{G}_m \supsetneq \bigcap_m \mathbf{G}_m = \mathbf{G}_\uparrow \supsetneq \mathbf{G}_\downarrow \supsetneq \mathbf{G}_\mathbb{R}$

1.4 Axioms and deduction systems for Gödel logics

In this section we introduce certain axioms and deduction systems for Gödel logics, and we will show completeness of these deduction systems subsequently.

NOTE: Most of the time we use Hilbert style systems, but for some proofs a Gentzen style (sequent) proof system will be adequate. In this proof system the notion of sequent, written as

$$A_1, \dots, A_n \Rightarrow B$$

is introduced which we will consider as an abbreviation for

$$A_1 \supset A_2 \supset \dots \supset A_n \supset B,$$

and $A_1, \dots, A_n \Rightarrow$ as an abbreviation for $A_1, \dots, A_n \Rightarrow \bot$.

We will denote by **IL** the complete axiom system for intuitionistic logic (taken from [Tro77]) given in Table 1.1.

The following axioms will play an important rôle:

QS	$\forall x(C^{(x)} \vee A(x)) \supset (C^{(x)} \vee \forall x A(x))$
LIN	$(A \supset B) \vee (B \supset A)$
ISO$_0$	$\forall x \neg\neg A(x) \supset \neg\neg\forall x A(x)$
ISO$_1$	$\forall x \neg \Delta A(x) \supset \neg\Delta\exists x A(x)$
FIN(n)	$(\top \supset p_1) \vee (p_1 \supset p_2) \vee \dots \vee (p_{n-2} \supset p_{n-1}) \vee (p_{n-1} \supset \bot)$

$$
\begin{array}{rl}
(I1), (MP) & \dfrac{A \quad A \supset B}{B} \\
(I2) & \dfrac{A \supset B \quad B \supset C}{A \supset C} \\
(I3) & A \vee A \supset A, A \supset A \wedge A \\
(I4) & A \supset A \vee B, A \wedge B \supset A \\
(I5) & A \vee B \supset B \vee A, A \wedge B \supset B \wedge A \\
(I6) & \dfrac{A \supset B}{C \vee A \supset C \vee B} \\
(I7) & \dfrac{A \wedge B \supset C}{A \supset (B \supset C)} \\
(I8) & \dfrac{A \supset (B \supset C)}{A \wedge B \supset C} \\
(I9) & \bot \supset A \\
(I10) & \dfrac{B^{(x)} \supset A(x)}{B^{(x)} \supset \forall x A(x)} \\
(I11) & \forall x A(x) \supset A(t) \\
(I12) & A(t) \supset \exists x A(x) \\
(I13) & \dfrac{A(x) \supset B^{(x)}}{\exists x A(x) \supset B^{(x)}}
\end{array}
$$

(where $B^{(x)}$ means that x is not free in B).

Table 1.1: Axiom system for **IL**

For the axiomatization of quantified propositional Gödel logics we use

$$
\begin{array}{ll}
\text{DEN} & \forall p(A^{(p)} \supset p \vee p \supset B^{(p)}) \supset (A^{(p)} \supset B^{(p)}) \\
\text{QS}_{qp} & \forall p(C^{(p)} \vee A(p)) \supset (C^{(p)} \vee \forall p A(p))
\end{array}
$$

For the axiomatization of the Δ-operator we use

$$
\begin{array}{ll}
\Delta 1 & \Delta A \vee \neg \Delta A \\
\Delta 2 & \Delta(A \vee B) \supset (\Delta A \vee \Delta B) \\
\Delta 3 & \Delta A \supset A \\
\Delta 4 & \Delta A \supset \Delta\Delta A \\
\Delta 5 & \Delta(A \supset B) \supset (\Delta A \supset \Delta B)
\end{array}
$$

We will refer to the Δ-axioms given above combined with the rule: From A deduce ΔA, as AXΔ.

NOTE: The names of the axioms can be explained as follows: QS stands for 'quantifier shift', LIN for 'linearity', ISO_0 for 'isolation axiom of 0', ISO_1 for 'isolation axiom of 1', $\text{FIN}(n)$ for 'finite with n elements' and DEN for 'density' (the axiomatization of Takeuti).

DEFINITION 1.16 *If $\mathcal{A}$ is an axiom system, we denote by $\mathcal{A}^0$ the propositional part of $\mathcal{A}$, i.e. all the axioms which do not contain quantifiers.*

With $\mathcal{A}\Delta$ we denote the axiom system obtained from $\mathcal{A}$ by adding the axioms and rules AXΔ.

With $\mathcal{A}_n$ we denote the axiom system obtained from $\mathcal{A}$ by adding the axiom $\text{FIN}(n)$.

We denote by $\mathbf{H}^{qp}$ *the axiom system* $\mathbf{IL} + \text{QS}_{qp} + \text{LIN} + \text{DEN}$.

We denote by $\mathbf{H}$ *the axiom system* $\mathbf{IL} + \text{QS} + \text{LIN}$.

EXAMPLE 1.17 $\mathbf{IL}^0$ is the same as **IPL**. $\mathbf{H}^0$ is the same as **LC**.

For all these axiom systems the general notion of deducability can be defined:

DEFINITION 1.18 *If a formula/sequent Γ can be deduced from an axiom system $\mathcal{A}$ we denote this by*

$$\vdash_{\mathcal{A}} \Gamma$$

NOTE: The $\mathcal{A}$ in $\vdash_{\mathcal{A}} \Gamma$ often is omitted when it is obvious from the context which axiom system is meant. The notion $\Pi \vdash_{\mathcal{A}} \Gamma$ for finite Π is equivalent to $\vdash_{\mathcal{A}\cup\Pi} \Gamma$.

In the case where Σ is infinite, provability of $\Sigma \Rightarrow \Delta$ is defined by the existence of a finite subset $\Sigma' \subset \Sigma$ such that $\Sigma' \Rightarrow \Delta$ is provable. Validity in $[0,1]$ is defined via the entailment, see Chapter 4.

1.5 Completeness results of H for $\mathbf{G}_{[0,1]}$

In the discussion of completeness of Gödel logics the term 'completeness' stands for the existence of a complete recursive axiomatization, i.e. an axiom system together with rules which can deduce all valid formulas of Gödel logics. In the following we often will use the term *completeness* when in fact we are referring to *complete recursive axiomatization.*

There have been various proofs of the completeness of first-order Gödel logic $\mathbf{G}_{[0,1]}$. The first one is from Horn [Hor69], where the weak completeness is proven. Horn called the described logic the logic with truth values in a linearly ordered Heyting algebra and used the axiom system **H** as given above.

Later on Takeuti and Titani introduced intuitionistic fuzzy logic **IFL** in [TT84] and have shown that the following system TT is strongly complete for **IFL**:

DEFINITION 1.19 (TAKEUTI AND TITANI'S SYSTEM TT) *The axioms and inference rules of TT are those of* **LJ**[2] *together with the following axioms:*

1. $\Rightarrow (A \supset B) \vee ((A \supset B) \supset B)$
2. $(A \supset B) \supset B \Rightarrow (B \supset A) \vee B$
3. $(A \wedge B) \supset C \Rightarrow (A \supset C) \vee (B \supset C)$
4. $A \supset (B \vee C) \Rightarrow (A \supset B) \vee (A \supset C)$
5. $\forall x (C^{(x)} \vee A(x)) \Rightarrow C^{(x)} \vee \forall x A(x)$
6. $\forall x A(x) \supset C \Rightarrow \exists x ((A(x) \supset D^{(x)}) \vee (D^{(x)} \supset C))$.

and the following extra inference rule:

$$\frac{\Gamma \Rightarrow A \vee (C \supset p) \vee (p \supset B)}{\Gamma \Rightarrow A \vee (C \supset B)}$$

where p is any propositional variable not occurring in the lower sequent.

NOTE: The extra inference rule given above has an interesting property: It forces the truth value set to be dense in itself. This cannot be achieved by formulas, exhibiting the difference in expressive power of rules versus formulas in Gödel logics.

Finally Takano [Tak87] has shown that there is a strong completeness for the system **H**, and that the system **H**, the system TT and the system TT^- obtained from TT by dropping the extra inference rule are all equivalent, i.e. they prove the same formulas. A syntactical proof of the elimination of the TT-rule was later given by Baaz and Zach in [BZ00]. Thus, we see that the logic **IFL**, the logic with truth values in a linearly ordered Heyting algebra, and Gödel logics on the real interval $[0,1]$ coincide.

We will present Takanos proof of the strong completeness of the system **H** in detail because we will extend it to the case of Gödel logics with Δ and to different truth value sets.

1.5.1 Takanos completeness proof revisited

Horn [Hor69] only proved the weak completeness of Gödel logic, i.e. a formula A is valid iff it is provable in the system **H**. Takeuti and Titani [TT84] proved the strong completeness of Gödel logic, i.e. a sequent $\Sigma \Rightarrow \Delta$ is valid iff it is provable in TT, even if Σ is infinite.

[2] **LJ** is the sequent style calculus for intuitionistic logic.

Theorem 1.20 (Strong completeness of Gödel logic [Tak87]) *A sequent $\Sigma \Rightarrow \Delta$ (where Σ can be infinite) is valid in $[0,1]$ iff it is provable in* **H**.

We will assume $\Sigma \Rightarrow \Delta$ unprovable and construct a model $\mathcal{I}$ in which $\Sigma \Rightarrow \Delta$ is not valid. We will assume that there are infinitely many individual free variables which do not occur in $\Sigma \Rightarrow \Delta$, and that Δ consists of one formula A. Let $\mathcal{T}$ and $\mathcal{F}$ be the sets of all terms and all formulas, respectively.

Lemma 1.21 *There exists a set $\mathcal{G}$ of formulas which satisfies the following conditions:*

1. *$\Sigma \subseteq \mathcal{G}$ and $A \notin \mathcal{G}$.*
2. *If $\vdash \mathcal{G} \Rightarrow B_1 \vee \cdots \vee B_n$, then $B_i \in \mathcal{G}$ for some i.*
3. *if $B(t) \in \mathcal{G}$ for every $t \in \mathcal{T}$, then $\forall x B(x) \in \mathcal{G}$.*

Proof: Let $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$. We define a pair $\mathcal{G}_n, \mathcal{H}_n$ of subsets of $\mathcal{F}$ as follows: Let $\mathcal{G}_1 = \Sigma$ and $\mathcal{H}_1 = \Delta = \{A\}$. Assume that $\mathcal{G}_n$ and $\mathcal{H}_n$ have already been defined. If $\vdash \mathcal{G}_n \Rightarrow \bigvee \mathcal{H}_n \vee F_n$ set $\mathcal{G}_{n+1} = \mathcal{G}_n \cup \{F_n\}$ and $\mathcal{H}_{n+1} = \mathcal{H}_n$. Otherwise set $\mathcal{G}_{n+1} = \mathcal{G}_n$ and $\mathcal{H}_{n+1} = \mathcal{H}_n \cup \{F_n, B(a)\}$ or $\mathcal{H}_{n+1} = \mathcal{H}_n \cup \{F_n\}$ according as F_n has the form $\forall x B(x)$ or not, where a is any individual free variable which does not occur in $\mathcal{G}_n \cup \mathcal{H}_n \cup \{F_n\}$.

It is obvious from qs that $\nvdash \mathcal{G}_n \Rightarrow \bigvee \mathcal{H}_n$ by induction on n, and $\bigcup_{n=1}^{\infty} \mathcal{H}_n = \mathcal{F} \setminus \bigcup_{n=1}^{\infty} \mathcal{G}_n$. So $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n$ is the required set. □

Define relations $\leq^\circ$ and $\equiv$ on $\mathcal{F}$ by

$$B \leq^\circ C \Leftrightarrow B \supset C \in \mathcal{G} \quad \text{and} \quad B \equiv C \Leftrightarrow B \leq^\circ C \wedge C \leq^\circ B.$$

Then $\leq^\circ$ is reflexive and transitive, since for every B, C and D, $\vdash \Rightarrow B \supset B$, and $\vdash B \supset C, C \supset D \Rightarrow B \supset D$, so $B \supset B \in \mathcal{G}$ and if $B \supset C \in \mathcal{G}$ and $C \supset D \in \mathcal{G}$ then $B \supset D \in \mathcal{G}$. Hence, $\equiv$ is an equivalence relation on $\mathcal{F}$. For every B in $\mathcal{F}$ we let $|B|$ be the equivalence class under $\equiv$ to which B belongs, and $\mathcal{F}/\!\equiv$ the set of all equivalence classes. Next we define the relation $\leq$ on $\mathcal{F}/\!\equiv$ by

$$|B| \leq |C| \Leftrightarrow B \leq^\circ C \Leftrightarrow B \supset C \in \mathcal{G}.$$

Lemma 1.22 *$\langle \mathcal{F}/\!\equiv, \leq \rangle$ is a countably linearly ordered structure with distinct maximal element $|A \supset A|$ and the minimal element $|\neg(A \supset A)|$.*

Proof: Since $\mathcal{F}$ is countably infinite, $\mathcal{F}/\!\equiv$ is countable. For every B and C, $\vdash \Rightarrow (B \supset C) \vee (C \supset B)$ by lin, and so either $B \supset C \in \mathcal{G}$ or $C \supset B \in \mathcal{G}$, hence $\leq$ is linear. For every B, $\vdash \Rightarrow B \supset (A \supset A)$ and $\vdash \Rightarrow \neg(A \supset A) \supset B$, and so $B \supset (A \supset A) \in \mathcal{G}$ and $\neg(A \supset A) \supset B \in \mathcal{G}$, hence $|A \supset A|$ and $|\neg(A \supset A)|$ are the maximal and minimal elements, respectively.

Since $\vdash (A \supset A) \supset \neg(A \supset A) \Rightarrow A$ and $A \notin \mathcal{G}$, $(A \supset A) \supset \neg(A \supset A) \notin \mathcal{G}$, so $|A \supset A| \neq |\neg(A \supset A)|$. □

We abbreviate $|A \supset A|$ by $\mathbf{1}$ and $|\neg(A \supset A)|$ by $\mathbf{0}$.

LEMMA 1.23 *The following properties hold in* $\langle \mathcal{F}/\equiv, \leq \rangle$*:*

1. $|B \wedge C| = \min\{|B|, |C|\}$.
2. $|B \vee C| = \max\{|B|, |C|\}$.
3. $|B \supset C| = \mathbf{1}$ *if* $|B| \leq |C|$, $|B \supset C| = |C|$ *otherwise.*
4. $|\neg B| = \mathbf{1}$ *if* $|B| = \mathbf{0}$; $|\neg B| = \mathbf{0}$ *otherwise.*
5. $|\exists x B(x)| = \sup\{|B(t)| : t \in \mathcal{T}\}$.
6. $|\forall x B(x)| = \inf\{|B(t)| : t \in \mathcal{T}\}$.
7. $|B| = \mathbf{1} \Leftrightarrow B \in \mathcal{G}$.

PROOF: ad 1. From $\vdash \Rightarrow B \wedge C \supset B$, $\vdash \Rightarrow B \wedge C \supset C$ and $\vdash D \supset B, D \supset C \Rightarrow D \supset B \wedge C$ for every D, it follows that $|B \wedge C| = \inf\{|B|, |C|\}$, from which 1. follows since $\leq$ is linear.

2. is proved similarly to 1.

ad 3. From $\vdash \Rightarrow (B \supset C) \wedge B \supset C$ and $\vdash D \wedge B \supset C \Rightarrow D \supset (B \supset C)$ for every D, it follows that $|B \supset C| = \max\{|D| : |D \wedge B| \leq |C|\}$. Hence, in view of 1., follows 3. since $\leq$ is linear.

ad 4. From $\vdash \Rightarrow \neg B \wedge B \supset \neg(A \supset A)$ and $\vdash D \wedge B \supset \neg(A \supset A) \Rightarrow D \supset \neg B$ for every D, it follows that $|\neg B| = \max\{|D| : |D \wedge B| = \mathbf{0}\}$. Hence in view of 1., follows 4. since $\leq$ is linear.

ad 5. Since $\vdash \Rightarrow B(t) \supset \exists x B(x)$, $|B(t) \leq |\exists x B(x)|$ for every $t \in \mathcal{T}$. On the other hand, for every D,

	$\lvert B(t)\rvert \leq \lvert D\rvert$	for every $t \in \mathcal{T}$
$\Leftrightarrow$	$B(t) \supset D \in \mathcal{G}$	for every $t \in \mathcal{T}$
$\Rightarrow$	$\forall x(B(x) \supset D) \in \mathcal{G}$	since 3
$\Rightarrow$	$\exists x B(x) \supset D \in \mathcal{G}$	since $\vdash \forall x(B(x) \supset D) \Rightarrow \exists x B(x) \supset D$
$\Leftrightarrow$	$\lvert\exists x B(x)\rvert \leq \lvert D\rvert$.	

Hence, 5. follows.

6. is proved similarly to 5.

ad 7. Since $\vdash (A \supset A) \supset B \Rightarrow B$ and $\vdash B \Rightarrow (A \supset A) \supset B$,

$$|B| = \mathbf{1} \Leftrightarrow |A \supset A| \leq |B| \Leftrightarrow (A \supset A) \supset B \in \mathcal{G} \Leftrightarrow B \in \mathcal{G}.$$

□

Takano now cites a proposition from Horns paper:

LEMMA 1.24 (HORN [HOR69], LEMMA 3.7) *If $\langle L, \leq \rangle$ is a countable linearly ordered structure with distinct maximal and minimal elements, then there exists a monomorphism from $\langle L, \leq \rangle$ to $\langle [0,1] \cap \mathbb{Q}, \leq \rangle$ which preserves the maximal and the minimal elements as well as all existing suprema and infima in $\langle L, \leq \rangle$. Hence, there exists such a monomorphism on $\langle L, \leq \rangle$ to $\langle [0,1], \leq \rangle$.*

PROOF: Let the members of L be arranged in a sequence: $a_0 = \mathbf{0}$, $a_1 = \mathbf{1}$, $a_2, \cdots$. Let $h(\mathbf{0}) = 0$, $h(\mathbf{1}) = 1$. We define $h(a_n)$ inductively: Let a_i be the largest member of $\{a_k : k < n\}$ which is $< a_n$, and let a_j be the smallest member which is $> a_n$. Then let

$$h(a_n) = \frac{h(a_i) + h(a_j)}{2}.$$

□

Horn concludes the proof with the comment that "It is not hard to verify that h has the required properties", and in fact most properties are trivial.

The really crucial point in this lemma is the preserving of infima and suprema. We want to exhibit this preserving in more detail:

LEMMA 1.25 *The evaluation defined by $\mathcal{I}(A) = h(|A|)$ is a valuation.*

PROOF: What in fact has to be proved is that $\mathcal{I}(\forall x A(x))$ and $\mathcal{I}(\exists x A(x))$ are 'well defined' in the sense, that the truth value of a quantified formula computed from the above definition coincides with the truth value computed via the distribution. For the propositional connectives this is trivial.

The truth value of $\forall x A(x)$ can be computed in two ways:

$$\mathcal{I}(\forall x A(x)) = h(\inf_L\{|A(t)| : t \in T\})$$
$$\mathcal{I}(\forall x A(x)) = \inf_{\mathbb{R}}\{h(|A(t)|) : t \in T\}$$

The first one uses the definition of the equivalence class, the second one the necessary properties of a valuation. We want to show that these two definitions coincide, which proves the lemma.

Let $L = F/\equiv = \{a_0 = \mathbf{0}, a_1 = \mathbf{1}, a_2, \ldots\}$ and $|\forall x A(x)| = a_N$, $h(a_N) = s$. Furthermore let

$$\inf_{\mathbb{R}}\{h(|A(t)|) : t \in T\} = q.$$

We assume that $s \neq q$, i.e. the interpretations do *not* coincide. So we have $s < q$.

As a first step it is easy to show that *all* a_i with $a_i > a_N$ are mapped into $(q, 1]$, i.e. $h(a_i) > q$. The proof is as follows: Assume that a_i is mapped into an element less than q. Since a_N is the $\inf_L$ we find a term t such that $a_N \leq |A(t)| < a_i$, which yields a contradiction using the fact that h is a homomorphism.

The final step is to show that there is an element a_u that is mapped into (s, q) which would yield a contradiction. For this let l be minimal (the first) such that

1. $l > N$
2. $\forall u < l : a_u < a_N \vee a_l < a_u$
3. $h(a_l) \in (q, q + \frac{q-s}{3})$.

Such an a_l exists because q is the infimum of $\{h(a_i) : a_i > a_N\}$. See Figure 1.1 for explanation.

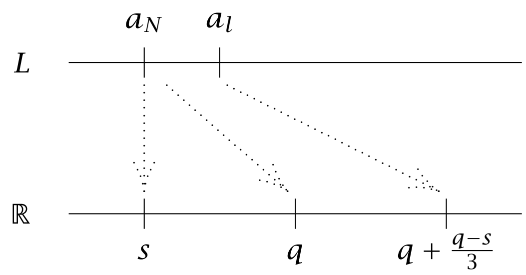

Figure 1.1: The mapping h from $\langle L, \leq\rangle$ to $\langle [0,1], \leq\rangle$

Now choose u minimal such that $u > l$ and $a_N < a_u < a_l$, which must exist since a_N is the infimum. We compute the value of a_u under h:

$$\begin{aligned} h(a_u) - s &= \frac{h(a_N) + \min\{h(a_i) : i < u \wedge a_i > a_u\}}{2} - s \\ &< \frac{s + (q + (q-s)/3)}{2} - s \\ &= \frac{s + 2q}{3} - s \\ &= \frac{2}{3}(q - s) \end{aligned}$$

Therefore, $h(a_u) \in (s, q)$ which is a contradiction.

The case of the existential quantifier is treated accordingly. □

We are now ready to give the final proof for Theorem 1.20:

Proof: [of Theorem 1.20] By the above lemmas there exists a monomorphism h from $\langle \mathcal{F}/\equiv, \leq\rangle$ into $\langle [0,1], \leq\rangle$ which preserves the maximal and the minimal elements as well as all existing suprema and infima in $\langle \mathcal{F}/\equiv, \leq\rangle$. Put $\mathcal{I}(B) = h(|B|)$ for every $B \in \mathcal{F}$ and we obtain a model.

Note that for every B,

$$\mathcal{I}(B) = 1 \Leftrightarrow |B| = \mathbf{1} \Leftrightarrow B \in \mathcal{G}.$$

In this model,

$$B \in \Sigma \Rightarrow B \in \mathcal{G} \Leftrightarrow \mathcal{I}(B) = 1,$$

while $A \notin \mathcal{G}$ so $\mathcal{I}(A) \neq 1$, so $\Sigma \Rightarrow \Delta$ is not valid.

Thus we have proven that on the assumption that $\Sigma \Rightarrow \Delta$ is unprovable, there is a model in which it is not valid. This is strong completeness. □

As we have already mentioned on p. 3 in Lemma 1.1 the definition of truth value for Gödel implication can be obtained from very simple properties, one of it being the existence of a deduction theorem.

COROLLARY 1.26 (DEDUCTION THEOREM FOR GÖDEL LOGICS) *Let T be a theory over* **H** *then*

$$T, A \vdash B \quad \textit{iff} \quad T \vdash A \supset B$$

In the first-order case the free variables of A and B must be disjoint.

PROOF: The deduction theorem for Gödel logics is an immediate consequence of Lemma 1.1 together with the completeness result. Another proof would be by induction on the length of the proof. See [Háj98], Theorem 2.2.18. □

1.6 Completeness results of $\mathbf{H}\Delta$ for $\mathbf{G}^{\Delta}_{[0,1]}$

We will now extend the proof given above to Gödel logics with Δ on $[0, 1]$. In fact we can leave the proof as it is, only changing the definition of the provable sequents from

$$\mathcal{G}_n \Rightarrow \mathcal{H}_n$$

to

$$\Delta\mathcal{G}_n \Rightarrow \mathcal{H}_n.$$

The only fact we have to prove is that the interpretation of the syntactical Δ behaves exactly like we want, i.e. like the function δ:

$$\delta(1) = 1 \qquad \delta(x) = 0 \text{ for } x < 1$$

So we have to proof the following lemma

LEMMA 1.27

$$h(|\Delta B|) = \delta(h(|B|))$$

PROOF: We will use the following fact: If $\Delta\mathcal{G} \Rightarrow A$ and $\Rightarrow A \supset B$ then $\Delta\mathcal{G} \Rightarrow B$. First assume that $\Delta B \in \mathcal{G}$, therefore, $|\Delta B| = \mathbf{1}$ and $h(|\Delta B|) = 1$, too. We want to show that $h(|B|) = 1$, because only then $\delta(h(|B|)) = 1$, too. Using the axiom

$$\Delta 3 \quad \vdash \Delta B \supset B$$

we obtain that $\Delta\mathcal{G} \Rightarrow B$, so $B \in \mathcal{G}$ which is equivalent to $|B| = \mathbf{1}$, therefore, $h(|B|) = 1$, which proves the first case.

Now assume that $\Delta B \notin \mathcal{G}$. First we compute the value of $h(|\Delta B|)$: The axiom

$$\Delta 1 \quad \vdash \Delta B \vee \neg\Delta B$$

together with the assumption $\Delta B \notin \mathcal{G}$ gives $\neg\Delta B \in \mathcal{G}$, therefore, $|\neg\Delta B| = \mathbf{1}$ and $|\Delta B| = \mathbf{0}$ and so

$$h(|\Delta B|) = 0.$$

Now we compute the value of $\delta(h(|B|))$: Assume that $B \in \mathcal{G}$. Using a variant of the Δ-introduction rule

$$\frac{\Delta\mathcal{G} \Rightarrow B}{\Delta\Delta\mathcal{G} \Rightarrow \Delta B}$$

which needs the axioms for shifting the Δ into implications and disjunctions ($\Delta 2$ and $\Delta 5$), together with the cancellation of double Δ by using $\Delta 4$ and modus ponens we obtain $\Delta B \in \mathcal{G}$, which is a contradiction. So we have shown that $B \notin \mathcal{G}$, but this is equivalent to $h(|B|) < 1$ and so $\delta(h(|B|)) = 0$, which proves the second part. □

Note that we have used all the axioms for Δ, including the introduction rule, which yields the following theorem:

THEOREM 1.28 (STRONG COMPLETENESS OF GÖDEL LOGIC WITH Δ) *A sequent $\Sigma \Rightarrow \Delta$ in the logic with Δ is valid in $[0,1]$ iff it is provable in* $\mathbf{H}\Delta$.

An interesting property is that there is no standard deduction theorem for Gödel logics with Δ as opposed to Gödel logics without Δ, where we can go from $A \vdash B$ to $\vdash A \supset B$. This is not the case for Gödel logics with Δ, as we cannot go from $A \vdash \Delta A$ to $\vdash A \supset \Delta A$. But we get a deduction theorem of the following form:

THEOREM 1.29 (DEDUCTION THEOREM FOR GÖDEL LOGICS WITH Δ) *Let T be a theory over* $\mathbf{H}\Delta$ *then*

$$T, A \vdash B \quad \textit{iff} \quad T \vdash \Delta A \supset B$$

In the first-order case the free variables of A and B must be disjoint.

PROOF: The proof is an easy induction on the length of the proof. See [Háj98], Theorem 2.4.14. □

CHAPTER 2

Linear orderings and topology

If we take a close look at Takano's proof, we can extract the necessary properties of the underlying truth value set such that the same completeness proof goes through. These properties in fact are defined by the possibility to construct an isomorphism from the linear order $\langle \mathcal{F}/\equiv, \leq \rangle$ into a sub-ordering of the truth value set, which preserves all infima and suprema.

From a more computational point of view it always has to be possible to select an element in the truth value set which is between two given ones, i.e. for all distinct $a < b$ in the truth value set we always have to find an element c such that $a < c < b$. This in fact is the definition of a dense linear order.

This suggests that the completeness proof of Takano can be extended to Gödel logics based on truth value sets which contain a dense linear order. We will see later that all truth value sets which comply to this and an additional condition with respect to 0, will in fact generate the same Gödel logic, identical to $\mathbf{G}_{\mathbb{R}}$. This exhibits a dilemma with respect to the extensional definition of Gödel logics as the set of valid formulas over a truth value set, where different truth value sets induce the same set of valid formulas, i.e. the same Gödel logic. We are aiming at a dual semantic characterization of Gödel logics, and topological and order theoretic properties will provide this classification. Thus, this chapter is devoted to topological and order theoretic preliminaries.

2.1 Dense linear orderings

We discuss the necessary properties of linear orderings, but we do not give a full account of this topic. A complete and very detailed introduction can be found in [Ros82].

Definition 2.1 (Dense linear orderings) *A linear ordering is called* dense *if for distinct element $a < b$ there is an element c such that $a < c < b$, i.e.*

$$\forall a \forall b (a < b \supset \exists c (a < c < b))$$

As we have seen in Takanos proof of the completeness of standard Gödel logics it is necessary to provide a linear order into which every other countable linear oder can be embedded. Already Cantor proved the following important result on dense linear orderings:

Lemma 2.2 *Let $\langle A, R \rangle$ be a countable dense linear ordering which has no first nor last element. Let $\langle B, S \rangle$ be an arbitrary countable linear ordering. Then $\langle B, S \rangle$ is isomorphic to a sub-ordering of $\langle A, R \rangle$.*

Proof: See [Ros82]. □

Another interesting property is the fact that a bounded dense linear sub-ordering, say a sub-ordering of $[0, 1]$, when completed in the topological sense, yields a set which is closed and contains only limit points and finitely many isolated points.

This is an instance of a more general notion, namely the notion of perfect sets in Polish space. We will now give a short introduction to perfect sets and come back to dense linear orderings later on. In the presentation we follow [Kec95], where all the proofs are given, if not otherwise indicated.

2.2 Perfect sets

All the following notations, lemmas, theorems are carried out within the framework of Polish spaces, which are separable, completely metrizable topological spaces. For our discussion it is only necessary to know that $\mathbb{R}$ is such a Polish space.

Definition 2.3 (limit point, perfect space, perfect set) *A* limit point *of a topological space is a point that is not isolated, i.e. for every open neighborhood U of x there is a point $y \in U$ with $y \neq x$. A space is* perfect *if all its points are limit points. A set $P \subseteq \mathbb{R}$ is* perfect *if it is closed and together with the topology induced from $\mathbb{R}$ is a perfect space.*

It is obvious that all (non-trivial) closed intervals are perfect sets, also all countable unions of (non-trivial) intervals. But all these sets generated from closed intervals have the property that they are 'everywhere dense', i.e. contained in the closure of their inner component. There is another very famous set which is perfect but is nowhere dense, the Cantor set:

EXAMPLE 2.4 (CANTOR SET) The set of all numbers in the unit interval which can be expressed in triadic notation only by digits 0 and 2 is called *Cantor set.*

A more intuitive way to obtain this set is to start with the unit interval, take out the open middle third and restart this process with the lower and the upper third. Repeating this you get exactly the Cantor set because the middle third always contains the numbers which contain the digit 1 in their triadic notation.

This set has a lot of interesting properties, the most important one is that it is a perfect set:

PROPOSITION 2.5 *The Cantor set is perfect.*

It is possible to embed the Cauchy space into any perfect space, which yields the next lemma:

LEMMA 2.6 *If X is a nonempty perfect Polish space, then the cardinality of X is $2^{\aleph_0}$ and therefore, all nonempty perfect subsets, too, have cardinality of the continuum.*

It is possible to obtain the following characterization of perfect sets:

PROPOSITION 2.7 (CHARACTERIZATION OF PERFECT SETS) *For any perfect set there is a unique partition of the real line into countably many intervals such that the intersections of the perfect set with these intervals are either empty, the full interval or isomorphic to the Cantor set.*

PROOF: See [Win99]. □

So we see that intervals and Cantor sets are prototypical for perfect sets and the basic building blocks of more complex perfect sets.

Every Polish space can be partitioned into a perfect kernel and a countable rest. This is the well known Cantor-Bendixon Theorem:

DEFINITION 2.8 (CONDENSATION POINT) *A point x in a topological space X is a* condensation point *if every open neighborhood of x is uncountable.*

Note that the condition for limit point is that every open neighborhood is infinite, but not necessarily uncountable.

Theorem 2.9 (Cantor-Bendixon) *Let X be a Polish space. Then X can be uniquely written as $X = P \cup C$, with P a perfect subset of X and C countable and open. The subset P is called the* perfect kernel *of X.*

As a corollary we obtain that any uncountable Polish space contains a perfect set, and therefore, has cardinality $2^{\aleph_0}$.

2.3 Cantor-Bendixon derivatives and ranks

Definition 2.10 ((iterated) Cantor-Bendixon derivative) *For any topological space X let*

$$X' = \{x \in X : x \text{ is limit point of } X\}.$$

We call X' the Cantor-Bendixon derivative *of X.*

Using transfinite recursion we define the iterated Cantor-Bendixon derivatives *X^α, α ordinal, as follows:*

$$\begin{aligned} X^0 &= X \\ X^{\alpha+1} &= (X^\alpha)' \\ X^\lambda &= \bigcap_{\alpha<\lambda} X^\alpha, \text{ if } \lambda \text{ is limit ordinal.} \end{aligned}$$

It is obvious that X' is closed, that X is perfect iff $X = X'$, and that (X^α) for α ordinal is a decreasing transfinite sequence of closed subsets of X.

Theorem 2.11 *Let X be a Polish space. For some countable ordinal α_0, $X^\alpha = X^{\alpha_0}$ for all $\alpha \geq \alpha_0$ and X^{α_0} is the perfect kernel of X.*

Thus, it is possible to obtain the perfect kernel in a more constructive way. This leads to the definition of the Cantor-Bendixon rank:

Definition 2.12 (Cantor-Bendixon rank) *For any Polish space X, the least ordinal α_0 as above is called the* Cantor-Bendixon rank *of X and is denoted by $|X|_{\mathrm{CB}}$. We will denote the perfect kernel of X with X^∞ or $X^{|X|_{\mathrm{CB}}}$.*

2.4 The structure of countable compact topological spaces

If the space X is countable then $X^\infty = \emptyset$, since every non-empty perfect set has at least cardinality of the continuum. Now it is possible to give a finer characterization of these countable sets by analyzing their structure under the Cantor-Bendixon derivatives.

Definition 2.13 (rank of an element, topological type of X) *Let X be a countable topological space. For any $x \in X$, we can define its (Cantor-Bendixon-)rank*

$$\mathrm{rg}(x) = \sup\{\alpha : x \in X^\alpha\}.$$

Thus, we also can define the rank of X equivalently by

$$|X|_{\mathrm{CB}} = \sup\{\mathrm{rg}(x) : x \in X\}.$$

We call

$$\tau(X) = (\alpha, n), \quad \textit{with } \alpha = \alpha(X) = |X|_{\mathrm{CB}}, \quad n = n(X) = |X^{|X|_{\mathrm{CB}}}|$$

the topological type *of X.*

2.5 Relation between dense linear orderings and perfect sets

Coming back to Section 1.5 where we stated that a completeness proof à la Takano can be carried out if the truth value set contains a dense linear sub-ordering. The necessary condition of a truth-value set being closed in $[0, 1]$ transforms the existence of a dense linear sub-ordering into the condition that the truth-value set is uncountable:

Lemma 2.14 *The completion of a dense linear sub-ordering of $\langle [0, 1], < \rangle$ contains a perfect set.*

Proof: The completion of the order-type η (this is the order-type of a dense linear ordering) is λ, which is the order-type of the continuum, thus, in combination with Theorem 2.9, a completion of a dense linear ordering is uncountable. See [Ros82], Theorem 2.32 ff. □

We also want to go the other way round and prove that any perfect set contains a dense linear sub-ordering.

Definition 2.15 (inner/boundary point of a perfect set) *A point $p \in P$ is called* inner point *if there is a sequence (a_n) and a sequence (b_n) in P such that (a_n) is strictly increasing, (b_n) is strictly decreasing and $\lim a_n = \lim b_n = p$. All other points are called* boundary *points.*

Lemma 2.16 *For the Cantor set all the numbers with infinitely many 0 and infinitely many 2 in the triadic notation are inner numbers. All others are boundary points.*

PROOF: The lemma is obvious for all points with only finite triadic notation and for those points ending in infinitely many digits 2. We only have to show that all points with infinitely many 0 and 2 are inner points. Let

$$a = 0.a_1 a_2 a_3 \dots a_k \dots$$

and $i_1, i_2, \dots$ all the indices with $a_{i_l} = 0$ and $j_1, j_2, \dots$ all the indices with $a_{j_{l'}} = 0$. Consider the following two sequences:

$${}^l b = 0.a_1 a_2 \dots a_{i_{l-1}} 2 a_{i_{l+1}} \dots$$

and

$${}^l c = 0.a_1 a_2 \dots a_{i_{l-1}} 0 a_{i_{l+1}} \dots$$

Clearly, all the ${}^l b$ are larger then a and all the ${}^l c$ are smaller then a. Moreover, the ${}^l b$ are a strictly decreasing sequence with limit a in the perfect set, likewise the ${}^l c$ are a strictly increasing sequence with limit a in the perfect set. This proves that a is an inner point. □

Now we can construct a countable dense subset which only consists of inner points.

LEMMA 2.17 *For any two inner points of a perfect set there is an inner point strictly between the two, i.e. for all x and y, $x < y$, x, y inner points, there exists a z such that $x < z < y$ and z is inner point.*

PROOF: According to Proposition 2.7 all parts of perfect sets are either isomorphic to an interval or to the Cantor set, thus, we will show this property for intervals and for the Cantor set only.

For intervals it is trivial, for Cantor set take an arbitrary point $z' \in P$ between x and y. This is indeed possible since x and y are both inner points. If z' is inner point set $z = z'$. If not, it is a boundary point of the Cantor set, thus, has only a finite triadic notation length, or the triadic notation finishes with only 2. Either

$$z' = a = 0.a_1 a_2 a_3 \dots a_n$$

or

$$z' = b = 0.b_1 b_2 b_3 \dots b_n 222222 \dots$$

We approximate a with elements ${}^k a$, and ${}^k a$ is generated from a by the concatenation of the triadic notation of a with $2k$ zeros and a sequence of $0202\dots$, thus,

$$0.a_1 a_2 a_3 \dots a_n 0202020202 \dots$$
$$0.a_1 a_2 a_3 \dots a_n 0002020202 \dots$$
$$0.a_1 a_2 a_3 \dots a_n 0000020202 \dots$$

Furthermore, we approximate b with elements ${}^k b$ generated from b by replacing the notation of b starting from the $2k$-th '2' with $0202\ldots$, thus,

$$0.b_1 b_2 b_3 \ldots b_n 02020202\ldots$$
$$0.b_1 b_2 b_3 \ldots b_n 22020202\ldots$$
$$0.b_1 b_2 b_3 \ldots b_n 22220202\ldots$$

It is obvious from the triadic notation of the elements ${}^k a$ and ${}^k b$ that they are inner points of the perfect set. Additionally, it is obvious that $\lim_{k\to\infty} {}^k a = a$ and $\lim_{k\to\infty} {}^k b = b$. Let $\varepsilon < \min\{z' - x, y - z'\}$ and let N be large enough that $|z' - {}^N a| < \varepsilon$ (or $|z' - {}^N b| < \varepsilon$), then set $z = {}^N a$ (or $z = {}^N b$). □

Lemma 2.18 *For any perfect set P there is a countable dense linear sub-ordering $P_{\mathbb{Q}}$ consisting only of inner points.*

Proof: Start with 0 and 1, which can be counted as inner points, and iterate the lemma from above to generate a set of inner points, which is dense in itself. □

Thus, we have proven the equivalence of the existence of a perfect subset and the existence of a dense linear sub-ordering of a truth-value set.

Theorem 2.19 *A truth-value set contains a dense linear sub-ordering if and only if it is uncountable.*

Proof: If the truth-value set is uncountable, it contains a perfect set (Theorem 2.9) and by Lemma 2.18 a countable dense linear sub-ordering. If a truth-value set contains a dense linear sub-ordering, it must also contain the completion of this sub-ordering (a truth-value set must be closed). This completion contains a perfect set (Lemma 2.14), thus, also the truth-value set contains a perfect set and is uncountable. □

CHAPTER 3

Propositional Gödel logics

Another approach to Gödel logics is via restricting the possible accessibility relations of Kripke models of intuitionistic logic. Two somehow reasonable restrictions of the Kripke structures are the restriction to constant domains and the restriction that the Kripke worlds are linearly ordered. One can now ask what sentences are valid in this restricted class of Kripke models. This question has been settled by Dummett in [Dum59] for the propositional case by adding to a complete axiomatization of intuitionistic logic the axiom of linearity

$$\text{LIN} \qquad (p \supset q) \vee (q \supset p)$$

As we have seen in Lemma 1.10 this logic can be viewed as $\mathbf{G}_{\downarrow}^{0}$ and therefore, as a subcase of Gödel logics.

Another interesting distinction between **LC** or $\mathbf{G}_{\downarrow}^{0}$ and other propositional Gödel logics is the fact that the entailment relation of **LC** is not compact, while the one corresponding to $\mathbf{G}_{\mathbb{R}}^{0}$ is, as we will see in the next chapter.

3.1 Summary of results

We will show the following results

V infinite	$\mathbf{LC} = \mathbf{H}^0, \mathbf{H}\Delta^0$ complete for the logic	
	Theorem 3.4, p. 32, Theorem 3.10, p. 35	
V finite (n)	$\mathbf{LC}_n = \mathbf{H}^0_n, \mathbf{H}\Delta^0_n$ complete for the logic	
	Theorem 3.9, p. 34, Theorem 3.11, p. 35	

Table 3.1: Results for propositional logic

3.2 Completeness of $\mathbf{H}^0$ for LC

In [Dum59], Dummett proved that a formula of propositional Gödel logic is valid in any infinite truth value set if it is valid in one infinite truth value set. Moreover, all the formulas valid in these sets are axiomatized by any axiomatization of intuitionistic propositional logic extended with the linearity axiom scheme $(p \supset q) \vee (q \supset p)$. It is interesting to note that p and q in the linearity scheme are propositional formulas. It is *not* enough to add this axiom for atomic p and q. For an axiom scheme only necessary for atomic formulas we have to use

$$((p \supset q) \supset p) \vee (p \supset (p \supset q))$$

to obtain completeness [BV98]. The proof given here is a simplified proof of the completeness of $\mathbf{H}^0$ taken from Horn [Hor69].

DEFINITION 3.1 *An algebra* $\mathbf{P} = \langle P, \cdot, +, \to, \mathbf{0}, \mathbf{1} \rangle$ *is a* Heyting algebra *if* $\langle P, \cdot, +, \mathbf{0}, \mathbf{1} \rangle$ *is a lattice with least element* $\mathbf{0}$*, largest element* $\mathbf{1}$ *and* $x \cdot y \leq z$ *iff* $x \leq (y \to z)$.

DEFINITION 3.2 *An* L-algebra *is a Heyting algebra in which*

$$(x \to y) + (y \to x) = \mathbf{1}$$

is valid for all x, y.

It is obvious that if we take L-algebras as our reference models for completeness, the proof of completeness is trivial. Generally, it is not very interesting to define algebras fitting to logics like a second skin, and then proving completeness with respect to this class (Ł-algebras, ...), without giving any connection to well known algebraic structures or already accepted reference models. In our case we want to show completeness with respect to the real interval $[0, 1]$ or one of its sub-orderings. More generally we aim at completeness with respect to chains, which are special Heyting algebras:

DEFINITION 3.3 *A* chain *is a linearly ordered Heyting algebra.*

Chains are exactly what we are looking for as every chain (with cardinality less or equal to the continuum) is isomorphic to a sub-ordering of the $[0,1]$ interval, and vice versa. Our aim is now to show completeness of the above axiomatization with respect to chains. Furthermore we will exhibit that the length of the chains for a specific formula can be bounded by the number of propositional variables in the formula. More precisely:

THEOREM 3.4 *A formula α is provable in $\mathbf{H}^0 = \mathbf{LC}$ if and only if it is valid in all chains with at most $n+2$ elements, where n is the number of propositional variables in α.*

PROOF: As usual we define the relation $\alpha \leq^\circ \beta$ equivalent to $\vdash \alpha \supset \beta$ and $\alpha \equiv \beta$ as $\alpha \leq^\circ \beta$ and $\beta \leq^\circ \alpha$. It is easy to verify that $\equiv$ is an equivalence relation. We denote $\alpha/\equiv$ with $|\alpha|$. It is also easy to show that with $|\alpha| + |\beta| = |\alpha \vee \beta|$, $|\alpha| \cdot |\beta| = |\alpha \wedge \beta|$, $|\alpha| \to |\beta| = |\alpha \supset \beta|$ the set $\mathcal{F}/\equiv$ becomes a Heyting algebra, and due to the linearity axiom it is also an L-algebra. Furthermore note that $|\alpha| = \mathbf{1}$ if and only if α is provable in $\mathbf{H}^0$ ($\mathbf{1} = |p \supset p|$, $|\alpha| = |p \supset p|$ gives $\vdash (p \supset p) \supset \alpha$ which in turn gives $\vdash \alpha$).

If our aim would be completeness with respect to L-algebras the proof would be finished here, but we aim at completeness with respect to chains, therefore, we will take a close look at the structure of $\mathcal{F}/\equiv$ as L-algebra. Assume that a formula α is given, which is not provable, we want to give a chain where α is not valid. We already have an L-algebra where α is not valid, but how to obtain a chain?

We could use the general result from Horn, that a Heyting algebra is an L-algebra if and only if it is a subalgebra of a direct product of chains ([Hor69], Theorem 1.2), but we will exhibit how to find explicitly a suitable chain. The idea is that the L-algebra $\mathcal{F}/\equiv$ describes all possible truth values for all possible orderings of the propositional variables in α. We want to make this more explicit:

DEFINITION 3.5 *We denote with*

$$C(\bot, p_{i_1}, \ldots, p_{i_n}, \top)$$

the chain with these elements and the ordering

$$\bot \leq p_{i_1} < \ldots < p_{i_n} \leq \top.$$

If C is a chain we denote with $|\alpha|_C$ the evaluation of the formula in the chain C.

LEMMA 3.6 *The L-algebra $\mathcal{F}/\equiv$ is a subalgebra of the following direct product of chains*

$$X = \prod_{i=1}^{n!} C(\bot, \pi_i(p_1, \dots, p_n), \top)$$

where π_i ranges over the set of permutations of n elements. We will denote $C(\bot, \pi_i(p_1, \dots, p_n), \top)$ with C_i.

PROOF: Define $\phi : \mathcal{F}/\equiv \to X$ as follows:

$$\phi(|\alpha|) = (|\alpha|_{C_1}, \dots, |\alpha|_{C_{n!}}).$$

We have to show that ϕ is well defined, is a homomorphism and is injective. First assume that $\beta \in |\alpha|$ but $\phi(|\alpha|) \neq \phi(|\beta|)$, i.e.

$$(|\alpha|_{C_1}, \dots, |\alpha|_{C_{n!}}) \neq (|\beta|_{C_1}, \dots, |\beta|_{C_{n!}})$$

but then there must be an i such that

$$|\alpha|_{C_i} \neq |\beta|_{C_i}.$$

Without loss of generality, assume that $|\alpha|_{C_i} < |\beta|_{C_i}$. From the fact that $|\alpha| = |\beta|$ we get $\vdash \beta \supset \alpha$. From this we get that $|\beta \supset \alpha|_{C_i} < \mathbf{1}$ and from $\vdash \beta \supset \alpha$ we get that $|\beta \supset \alpha|_{C_i} = \mathbf{1}$, which is a contradiction. This proves the well-definedness.

To show that ϕ is a homomorphism we have to prove that

$$\phi(|\alpha| \cdot |\beta|) = \phi(|\alpha|) \cdot \phi(|\beta|)$$
$$\phi(|\alpha| + |\beta|) = \phi(|\alpha|) + \phi(|\beta|)$$
$$\phi(|\alpha| \to |\beta|) = \phi(|\alpha|) \to \phi(|\beta|).$$

This is a straightforward computation using $|\alpha \wedge \beta|_C = \phi(|\alpha|_C) \cdot \phi(|\beta|_C)$.

Finally we have to prove that ϕ is injective. Assume that $\phi(|\alpha|) = \phi(|\beta|)$ and that $|\alpha| \neq |\beta|$. From the former we obtain that $|\alpha|_{C_i} = |\beta|_{C_i}$ for all $1 \leq i \leq n!$, which means that

$$\mathcal{I}_{C_i}(\alpha) = \mathcal{I}_{C_i}(\beta) \quad \text{for all } 1 \leq i \leq n!.$$

On the other hand we know from the latter that there is an interpretation $\mathcal{I}$ such that $\mathcal{I}(\alpha) \neq \mathcal{I}(\beta)$. Without loss of generality assume that

$$\bot \leq \mathcal{I}(p_{i_1}) < \dots < \mathcal{I}(p_{i_n}) \leq \top.$$

There is an index k such that the C_k is exactly the above ordering with

$$\mathcal{I}_{C_k}(\alpha) \neq \mathcal{I}_{C_k}(\beta),$$

this is a contradiction.

This completes the proof that $\mathcal{F}/\equiv$ is a subalgebra of the given direct product of chains. □

EXAMPLE 3.7 For $n = 2$ the chains are $C(\bot, p, q, \top)$ and $C(\bot, q, p, \top)$. The product of these two chains looks as given in Figure 3.1, p. 34. The labels below the nodes are the products, the formulas above the nodes are representatives for the class $\alpha/\equiv$.

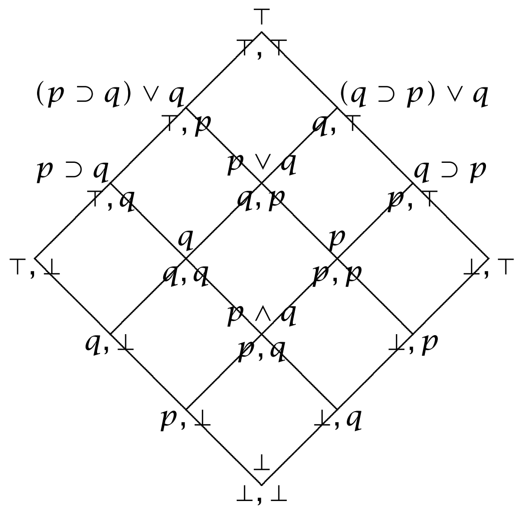

Figure 3.1: L-algebra of $C(\bot, p, q, \top) \times C(\bot, q, p, \top)$. Labels below the nodes are the elements of the direct product, formulas above the node are representatives for the class $\alpha/\equiv$.

Now the proof of Theorem 3.4 is trivial since, if $|\alpha| \neq \mathbf{1}$, there is a chain C_i where $|\alpha|_{C_i} \neq \mathbf{1}$. □

This yields the following theorem:

THEOREM 3.8 *A propositional formula is valid in any infinite chain iff it is derivable in* $\mathbf{LC} = \mathbf{H}^0$.

Going on to finite truth value set we can give the following theorem:

THEOREM 3.9 *A formula is valid in any chain with at most n elements iff it is provable in* $\mathbf{LC}_n$.

PROOF: Assuming that $\mathbf{H}^0_n \nvdash \alpha$ and using Corollary 1.26 we can proceed as follows:

$$\mathbf{H}^0_n \nvdash \alpha$$
$$\mathbf{H}^0 + \text{FIN}(n) \nvdash \alpha$$
$$\mathbf{H}^0 \nvdash \text{FIN}(n) \supset \alpha$$

From this we know that there is an interpretation $\mathcal{I}$ such that

$$\mathcal{I}(\text{FIN}(n) \supset \alpha) < 1$$

which is equivalent to

$$\mathcal{I}(\text{FIN}(n)) = 1 \text{ and } \mathcal{I}(\alpha) < 1.$$

The first formula ensures that the domain has at most n elements. Therefore, $\mathcal{I}$ is an interpretation with a domain with at most n elements and which evaluates α to a value less than 1. □

We can extend the above prove to the case with Δ:

THEOREM 3.10 *A propositional formula with Δ is valid in any infinite chain iff it is derivable from* $\mathbf{H}\Delta^0$.

PROOF: The proof is analogous to the case without Δ, the only point which needs attention is the well-definedness of the Δ, i.e. whether the interpretation of Δ defined in the proof really coincides with the intended interpretation. For this we have to show that $|\Delta A| = 1 \leftrightarrow |A| = 1$, and that either $|\Delta A|$ or $|\neg\Delta A|$ is equal to one. The former is easy to prove from the axioms, the latter is an immediate consequence of axiom $\Delta 1$. □

The same approach can be used to prove

THEOREM 3.11 *A propositional formula with Δ is valid in any chain with at most n elements iff it is provable in* $\mathbf{H}\Delta^0_n$.

3.3 Quantified propositional Gödel logics

In classical logic, propositional quantification does not increase expressive power per se. It does, however, allow to express complex properties more naturally and succinctly, e.g., in a sense satisfiability and validity of formulas are easily expressible within the logic. In contrast to classical propositional logic, propositional quantification may increase the expressive power of Gödel logics. More precisely, statements about the topological structure of the set of truth-values can be expressed using propositional quantifiers [BV98]. For example, the truth-value sets $[0, \frac{1}{2}] \cup \{1\}$ and $[0, 1]$, induce two different quantified propositional Gödel logics, but only one first-order Gödel logic.

THEOREM 3.12 *The system* $\mathbf{H}^{qp}$ *is sound and complete for* $\mathbf{G}^{qp}_\infty$.

PROOF: The proof can be found in [BV98]. □

Some other results which may be of interest in this area:

THEOREM 3.13 *Validity in* $\mathbf{G}_{\downarrow}^{qp}$ *is decidable.*

PROOF: The proof identifies a truth value $1/n$ with the infinite binary sequence $0^{n-1}1^{\omega}$, 0 with 0^{ω}, and uses a translation into S1S, see [BV98]. □

The interesting point in the following theorem is the existence of a quantifier elimination procedure, which was necessary to show that $\mathbf{G}_{\uparrow}^{qp}$ is the intersection of all the finitely valued quantified propositional Gödel logics $\mathbf{G}_{n}^{qp}$.

THEOREM 3.14 *The Gödel logic* $\mathbf{G}_{\uparrow}^{qp}$ *admits quantifier elimination, its validity is decidable and it is equal to the intersection of all finite-valued propositional Gödel logics.*

PROOF: The proof also uses a translation into S1S and can be found in [BCZ00]. □

Another very interesting property of quantified propositional Gödel logics is

THEOREM 3.15 *The number of quantified propositional Gödel logics is uncountable.*

PROOF: It is possible to express order and topological notions like being a boundary point of the truth value set, and ordering parts of the truth value set. This can be used for proving that there are uncountably many different logics. See [BV98] for details. □

CHAPTER 4

Propositional entailment

A question related to the completeness or recursive axiomatizablity of logics is on the compactness of the entailment relation of this logic. It has become increasingly obvious that such a study is called for, especially in cases where many-valued logics are applied in computer science to reasoning about various domains. For instance, Avron [Avr91] has argued that Gödel logics are suited to formalize properties of concurrency and advocated a view of logics primarily as entailment relations.

DEFINITION 4.1 (ENTAILMENT) *A (possibly infinite) set* Π *of formulas entails a formula* A *iff for all interpretations* $\mathcal{I}$ *the infimum of all the interpretations of formulas of* Π *is less or equal to the interpretation of* A*, i.e.*

$$\Pi \Vdash A \Leftrightarrow \forall \mathcal{I} \inf\{\mathcal{I}(B) : B \in \Pi\} \leq \mathcal{I}(A)$$

DEFINITION 4.2 (COMPACTNESS) $\mathbf{G}^0_V$ *is* compact *if, whenever* $\Pi \Vdash_V A$ *there is a finite* $\Pi' \subset \Pi$ *such that* $\Pi' \Vdash_V A$*.*

NOTE: It is important to mention that if we consider entailment relations or compactness, the underlying truth value set has to be closed under infima.

4.1 Summary of results

For propositional logic we will show that

V finite	$\mathbf{G}_V^0$ compact	Theorem 4.5, p. 38
V countable	$\mathbf{G}_V^0$ not compact	Theorem 4.7, p. 40
V uncountable	$\mathbf{G}_V^0$ compact	Theorem 4.6, p. 39

Table 4.1: Results for propositional entailment

In the case of propositional tautologies, all logics of infinite truth value sets are the same (cf. Chapter 3). The case for the entailment relation is similar with dense linear subset taking the position of the infinite subset.

One might wonder whether a different definition of the entailment relation in Gödel logic might give different results. Another standard way of defining entailment in many-valued logics is:

$$\Pi \Vvdash A \quad \text{iff} \quad \text{for all } \mathcal{I},\ (\forall B \in \Pi)(\mathcal{I}(B) = 1) \Rightarrow \mathcal{I}(A) = 1$$

This definition yields the same results, as the following proposition shows, allowing us to use the characterization of $\Vvdash$ or $\Vdash$ as convenient.

Proposition 4.3 *$\Pi \Vvdash_V A$ iff $\Pi \Vdash_V A$*

Proof: See [BZ98], Proposition 2.2 □

It is an easy but fundamental result that $\mathrm{Taut}(V) = \mathbf{G}_V^0$ and $\mathrm{Ent}(V)$, the set of valid entailment relations, depend only on the order type of V. This central property of Gödel logics is dependent on the specific definition of the Gödel implication, other definitions of implication might not allow this kind of equivalence. The following proposition makes this statement precise:

Proposition 4.4 *Let p and q propositional variables, A an arbitrary formula, $\mathcal{I}$ and $\mathcal{I}'$ valuations, not necessarily on the same sets of truth values, such that $\mathcal{I}(p) = 1$ iff $\mathcal{I}'(p) = 1$, $\mathcal{I}(p) < \mathcal{I}(q)$ iff $\mathcal{I}'(p) < \mathcal{I}'(q)$, and $\mathcal{I}(p) = \mathcal{I}(q)$ iff $\mathcal{I}'(p) = \mathcal{I}'(q)$ (for all p, q). Then $\mathcal{I}(A) = 1$ iff $\mathcal{I}'(A) = 1$ and $\mathcal{I}(A) = \mathcal{I}(p)$ iff $\mathcal{I}'(A) = \mathcal{I}'(p)$.*

Proof: The proof is straightforward by induction on the complexity of A. □

4.2 Finite truth value sets

Theorem 4.5 *If V is finite then $\mathbf{G}_V$ is compact.*

PROOF: We are discussing the entailment $\Pi \Vdash A$. Let $\Pi = \{B_1, B_2, \dots\}$, and let $X = \{p_0, p_1, \dots\}$, be an enumeration of variables occurring in Π, A such that all variables in B_i occur before the variables in B_{i+1}. We show that either $\{B_1, \dots, B_k\} \Vdash A$ for some $k \in \mathbb{N}$ or $\Pi \nVdash A$.

Let T be the complete semantic tree on X, i.e. $T = V^{<\omega}$. An element of T of length k is a valuation of $p_0, \dots, p_{k-1}$. Since V is finite, T is finitary. Let T' be the subtree of T defined by: $v \in T'$ if for every initial segment v' of v and every k such that all the variables in $A, B_1, \dots, B_k$ are among $p_0, \dots, p_{\ell(v')}$,

$$v'(\{B_1, \dots, B_k\}) = \min\{v(B_1), \dots, v(B_k)\} > v'(A).$$

In other words, branches in T' terminate at nodes v', where

$$v'(\{B_1, \dots, B_k\}) \leq v(A).$$

Now if T' is finite, there is a k such that $B_1, \dots, B_k \Vdash_V A$. Otherwise, since T' is finitary, it contains an infinite branch. Let v be the limit of the partial valuations in that branch. Obviously, since V is finite, $v(\Pi) > v(A)$ and so $\Pi \nVdash_V A$. □

4.3 Uncountable truth value sets

THEOREM 4.6 *If V is uncountable, then $\mathbf{G}_V$ is compact.*

PROOF: Let W be a densely ordered, countable subset of V. Such a subset exists according to Theorem 2.19. Let X be a set of variables. A *chain on* X is an arrangement of X in a linear order. Formally, a chain C on X is a sequence of pairs $\langle p_i, o_i \rangle$ where $o_i \in \{<, =, >\}$ where p_i appears exactly once. A valuation $\mathcal{I}$ respects C if $\mathcal{I}(p_i) = \mathcal{I}(p_{i+1})$ if o_i is $=$, $\mathcal{I}(p_i) > \mathcal{I}(p_{i+1})$ if o_i is $>$, and $\mathcal{I}(p_i) < \mathcal{I}(p_{i+1})$ if o_i is $<$. If X is finite, there are only finitely many chains on X.

We consider the entailment relation $\Pi \Vdash A$ and construct a tree in stages as follows: The initial node is labeled by $0 < 1$ and an empty valuation. Stage $n+1$: A node N constructed in stage n is labeled by a chain on the variables $p_1, \dots, p_n$ and a valuation $\mathcal{I}_N$ of $p_1, \dots, p_n$ respecting the chain. N receives successor nodes, one for each possibility of extending the chain by inserting p_{n+1}. The labels of each successor node N' are the corresponding extended chain and an extension of $\mathcal{I}_N$ which respects the extended chain. The value $\mathcal{I}_{N'}(p_{n+1})$ is chosen inside W, i.e. the endpoints of W may not be chosen as values. Since W is densely ordered, this ensures that such a choice can be made at every stage.

We call a branch of T *closed at node* N (constructed at stage n) if for some finite $\Pi' \subseteq \Pi$ such that $\mathrm{var}(\Pi') \cup \mathrm{var}(A) \subseteq \{p_1, \dots, p_n\}$ it holds that $\mathcal{I}_N(\Pi') \le \mathcal{I}_N(A)$. T is *closed* if it is closed on every branch. In that case, for some finite $\Pi' \subseteq \Pi$, we have $\Pi' \Vdash A$.

If T is not closed, it contains an infinite branch. Let $\mathcal{I}$ be the limit of the $\mathcal{I}_N$ of nodes N on the infinite branch. It holds that $\mathcal{I}(B) > \mathcal{I}(A)$ for all $B \in \Pi$, for otherwise the branch would be closed at the first stage where all the variables in A were assigned values. Let $w = \mathcal{I}(A)$. By Lemma 1.14, $\mathcal{I}_w(A) = \mathcal{I}(A)$ and $\mathcal{I}_w(\Pi) = \inf\{\mathcal{I}_w(B) : B \in \Pi\} = 1$, and so $\Pi \nVdash A$, a contradiction. □

4.4 Countable truth value sets

THEOREM 4.7 *If V is countably infinite, then $\mathbf{G}_V$ is not compact.*

PROOF: Note that if V is countable, it cannot contain a densely ordered subset, since truth-value sets for entailment have to be closed (under infima). We define a sequence of formulas Γ_k as follows:

$$\begin{aligned} \Gamma'_k &= \{p_{0/2^k} \prec p_{1/2^k} \prec \ldots \prec p_{(2^k-1)/2^k} \prec p_{2^k/2^k}\} \\ \Gamma''_k &= \{p_{0/2^k} \supset q, \ldots, p_{2^k/2^k} \supset q\} \\ \Gamma_k &= \Gamma'_k \cup \Gamma''_k \\ \Gamma &= \bigcup_{k\in\omega} \Gamma_k \end{aligned}$$

Intuitively, Γ'_k expresses that the $p_r = p_{i/2^k}$ are linearly ordered and $\bigcup_{k\in\omega} \Gamma'_k$ expresses that the variables p_r are densely ordered. Since V does not contain a densely ordered subset, we have

$$\Gamma \Vdash_V q.$$

In fact the only $\mathcal{I}$ such that $\mathcal{I}(\Gamma) = 1$ is $\mathcal{I}(p_r) = 1$ for all r, and $\mathcal{I}(q) = 1$. Now assume a finite $\Gamma' \subset \Gamma$ such that

$$\Gamma' \Vdash_V q.$$

There is a $\Gamma_k \supseteq \Gamma'$. Since V is infinite we can choose at least $2^k + 2$ different truth values $v_0 < \cdots < v_{2^k+1} < 1$. Define the valuation $\mathcal{I}$ as

$$\begin{aligned} \mathcal{I}(p_{i/2^k}) &= v_i \\ \mathcal{I}(q) &= v_{2^k+1}. \end{aligned}$$

Then we have $\mathcal{I}(\Gamma_k) = \mathcal{I}(\Gamma') = 1$, but $\mathcal{I}(q) < 1$ and therefore, $\Gamma' \nVdash_V q$. □

Thus, we have succeeded in characterizing the compact propositional Gödel logics. They are all those where the set of truth values V is either finite or contains a nontrivial densely ordered subset.

NOTE: Although the question whether there are uncountably many *first-order Gödel logics* has not been answered till now, it is possible to prove that there are uncountably ($2^{\aleph_0}$) many entailment relations, by expressing ordinals and their ordering. There are uncountably many such orderings, thus, also uncountably many different entailment relations[1].

[1]Martin Goldstern, oral communication

CHAPTER 5

First-order Gödel logics

5.1 Summary of results

We will derive the results as listed in Table 5.1 and Table 5.2.

V finite (n)	$\mathbf{H}_n$ complete for the logic Theorem 5.1, p. 42
V countable	not recursively enumerable Theorem 5.3, p. 43
$V^\infty \neq \emptyset$, $0 \in V^\infty$	$\mathbf{H}$ complete for the logic Theorem 5.5, p. 49
$V^\infty \neq \emptyset$, 0 isolated	$\mathbf{H} + \text{ISO}_0$ complete for the logic Theorem 5.6, p. 50
$V^\infty \neq \emptyset$, $0 \notin V^\infty$, 0 not isolated	not recursively enumerable Theorem 5.8, p. 52

Table 5.1: Results for first-order logic

5.2 Finite truth value sets

THEOREM 5.1 *A formula of first-order language is valid in all chains with at most n elements iff it is provable in* $\mathbf{H}_n$.

V finite (n)	$\mathbf{H}\Delta_n$ complete for the logic Theorem 5.2, p. 43
$0, 1 \in V^\infty$	$\mathbf{H}\Delta$ complete for the logic Theorem 5.9, p. 55
$0 \in V^\infty$, 1 isolated	$\mathbf{H}\Delta + \textsc{iso}_1$ complete for the logic Theorem 5.10, p. 55
0 isolated, $1 \in V^\infty$	$\mathbf{H}\Delta + \textsc{iso}_0$ complete for the logic Theorem 5.10, p. 55
$V^\infty \neq \emptyset$, 0, 1 isolated	$\mathbf{H}\Delta + \textsc{iso}_0 + \textsc{iso}_1$ complete Theorem 5.10, p. 55
$V^\infty \neq \emptyset$, $1 \notin V^\infty$, 1 not isolated	not recursively enumerable Theorem 5.8, p. 52

Table 5.2: Results for first-order logic with Δ

THEOREM 5.2 *A formula of first-order language with Δ is valid in all chains with at most n elements iff it is provable in* $\mathbf{H}\Delta_n$.

PROOF: The proofs are analogous to the proof of Theorem 3.9, p. 34. □

5.3 Countable truth value sets

In this section we prove that Gödel logics with countable truth value sets, i.e. those Gödel logics where the truth value set does not contain a dense subset, are not axiomatizable. We establish the result by reducing the classical validity of a formula in all finite models to the validity of a formula in Gödel logic. This yields the non-axiomatizability due to Trachtenbrot's theorem [Tra50].

THEOREM 5.3 *If V is countably infinite, then* $\mathbf{G}_V$ *is not axiomatizable.*

PROOF: We use the following approach: For every sentence A there is a sentence A^g s.t. A^g is valid in $\mathbf{G}_V$ iff A is true in every finite (classical) first-order structure. By Theorem 2.6, V is countably infinite iff it does not contain an infinite densely ordered subset.

We define A^g as follows: Let P be a unary and L be a binary predicate symbol not occurring in A and let $Q_1, \ldots, Q_n$ be all the predicate symbols in A. We use the following abbreviations:

$$x \in y \equiv \neg\neg L(x, y)$$
$$x \prec y \equiv P(x) \prec P(y) \equiv (P(y) \supset P(x)) \supset P(y)$$

Note that for any interpretation $\mathcal{I}$, $\mathcal{I}(x \in y)$ is either 0 or 1, and as long as $\mathcal{I}(P(x)) < 1$ (in particular, if $\mathcal{I}(\exists z P(z)) < 1$), we have $\mathcal{I}(x \prec y) = 1$

iff $\mathcal{I}(P(x)) < \mathcal{I}(P(y))$. Let

$$A^g \equiv S \wedge c_1 \in 0 \wedge c_2 \in 0 \wedge c_2 \prec c_1 \wedge \\ \wedge \forall i[\forall x, y \forall j, k \exists z\, Levels \vee \forall x \neg(x \in s(i))] \supset (A' \vee \exists u\, P(u))$$

where S is the conjunction of the standard axioms for 0, successor s and $\leq$, with double negations in front of atomic formulas,

$$Levels \equiv \begin{array}{c} j \leq i \wedge x \in j \wedge k \leq i \wedge y \in k \wedge x \prec y \supset \\ \supset (z \in s(i) \wedge x \prec z \wedge z \prec y) \end{array}$$

and A' is A where every atomic formula is replaced by its double negation, and all quantifiers are relativized to the predicate $R(i) \equiv \exists x(x \in i)$.

Intuitively, L is a predicate that divides a subset of the domain into levels, and $x \in i$ means that x is an element of level i. P orders the elements of the domain which fall into one of the levels in a sub-ordering of the truth values. The idea is that for any two elements in a level $\leq i$ there is an element in level $i+1$ which lies strictly between those two elements in the ordering given by $\prec$. If this condition cannot be satisfied, the levels above i are empty. Clearly, this condition can be satisfied only for finitely many levels unless V contains a dense subset, since if more than finitely many levels are non-empty, then $\bigcup_i \{x : x \in i\}$ gives a dense subset. By relativizing the quantifiers in A to the indices of non-empty levels, we in effect relativize to a finite subset of the domain. We make this more precise:

Suppose A is classically false in some finite structure $\mathcal{I}$. Without loss of generality, we may assume that the domain of this structure is the naturals $0, \ldots, n$. We extend $\mathcal{I}$ to a $\mathbf{G}_{\mathbb{R}}$ interpretation $\mathcal{I}^g$ with domain $\mathbb{N}$ as follows: Since V contains infinitely many values, we can choose c_1, c_2, L and P so that $\exists x(x \in i)$ is true for $i = 0, \ldots, n$ and false otherwise, and so that $\sup \mathrm{Distr}_{\mathcal{I}^g} P(x) < 1$. The number theoretic symbols receive their natural interpretation. The antecedent of A^g clearly receives the value 1, and the consequent receives $\sup \mathrm{Distr}_{\mathcal{I}^g} P(x) < 1$, so $\mathcal{I}^g \not\models A^g$.

Now suppose that $\mathcal{I} \not\models A^g$. Then $\mathcal{I}(\exists x\, P(x)) < 1$ and $\sup \mathrm{Distr}_{\mathcal{I}} P(x) < 1$. In this case, $\mathcal{I}(x \prec y) = 1$ iff $\mathcal{I}(P(x)) < \mathcal{I}(P(y))$, so $\prec$ defines a strict order on the domain of $\mathcal{I}$. It is easily seen that in order for the value of the antecedent of A^g under $\mathcal{I}$ to be greater than that of the consequent, it must be $= 1$ (the values of all subformulas are either $\leq \sup \mathrm{Distr}_{\mathcal{I}} P(x)$ or $= 1$). For this to happen, of course, what the antecedent is intended to express must actually be true in $\mathcal{I}$, i.e. that $x \in i$ defines a series of disjoint levels and that for any i, either level i is empty or for all x, y s.t. $x \in j$, $y \in k$ with $j, k \leq i$ and $x \prec y$ there is a z with $x \prec z \prec y$ and $z \in i+1$. To see this, consider the relevant part of the antecedent,

$$B = \forall i[\forall x, y \forall j \forall k \exists z\, Levels \vee \forall x \neg(x \in s(i))].$$

If $\mathcal{I}(B) = 1$, then for all i, either

$$\mathcal{I}(\forall x, y \forall j \forall k \exists z\, Levels) = 1$$

or $\mathcal{I}(\forall x \neg(x \in s(i))) = 1$. In the first case, we have

$$\mathcal{I}(\exists z\, Levels) = 1$$

for all x, y, j, and k. If it is not the case that for some z, $\mathcal{I}(Levels) < 1$, yet $\mathcal{I}(\exists z\, Levels) = 1$. Then, for at least some z the value of that formula would have to be $> \sup \mathrm{Distr}_{\mathcal{I}} P(z)$, which is impossible. Thus, for every x, y, j, k, there is a z such that $\mathcal{I}(Levels) = 1$. But this means that for all x, y s.t. $x \in j$, $y \in k$ with $j, k \leq i$ and $x \prec y$ there is a z with $x \prec z \prec y$ and $z \in i + 1$.

In the second case, where $\mathcal{I}(\forall x \neg(x \in s(i))) = 1$, we have that $\mathcal{I}(\neg(x \in s(i))) = 1$ for all x, hence $\mathcal{I}(x \in s(i)) = 0$ and level $s(i)$ is empty.

Since V contains no dense subset, from some finite level i onward, the levels must be empty. Of course, $i > 0$ since $c_1 \in 0$. Thus, A is false in the classical interpretation $\mathcal{I}^c$ obtained from $\mathcal{I}$ by restricting $\mathcal{I}$ to the domain $\{0, \ldots, i-1\}$ and $\mathcal{I}^c(Q) = \mathcal{I}(\neg\neg Q)$ for atomic Q. □

This shows that no infinite-valued Gödel logic which does not contain a dense subset is axiomatizable. Furthermore it has been shown in [Baa96] that all the Gödel logics based on truth value sets with topological type $\tau = (1, n)$, i.e. with Cantor-Bendixon rank of 1 and n limit points, are distinct. A similar result has been obtained in [Pre02] for Gödel logics with truth value sets of topological type $\tau = (n, 1)$, i.e. Cantor-Bendixon rank of n. This exhibits that there are at least countably many different first-order Gödel logics. The question whether there are uncountably many first-order Gödel logics still awaits settling.

5.4 Uncountable truth value sets

In the light of the previous chapters we will first revisit Takano's proof given on p. 17. The crucial fact, the one dependent on the truth value set, is the function h mapping $\langle \mathcal{F}/\!\equiv, \leq \rangle$ via $\langle [0,1] \cap \mathbb{Q}, \leq \rangle$ to $\langle [0,1], \leq \rangle$. This function has to be order preserving and infima and suprema preserving. Using Lemma 2.18 we can find a countable dense linear sub-ordering in every perfect set. Unfortunately, we cannot use Lemma 2.2 directly since we do not know if all infima and suprema are preserved.

Another detail we have to observe is the position of the countable dense linear sub-ordering. This is not arbitrary, since we can express a descent to 0 in the logical language:

$$\forall x \neg\neg A(x) \wedge \neg \forall x A(x)$$

The meaning of this formula is that all values of $A(x)$ are greater than 0, but the value of $\forall x A(x)$ is 0, thus, providing an infinite decreasing sequence to 0.

As a consequence, the placement of the countable dense linear subset is not arbitrary, but must have 0 as smallest element. This extends to the perfect set which must contain 0, too. If these conditions are fulfilled we still have to find a function h as above.

5.4.1 Case $0 \in V^\infty$

We want to extend Takanos proof such that it proves the completeness of any Gödel logics with a truth value set which contains a perfect set which in turn contains 0. For this we have to extend Horn's lemma from p.19 such that we can give a monomorphism from $\langle L, \leq \rangle$ to a countable dense sub-ordering of the perfect set V^∞, call it $P_\mathbb{Q}$, so we need a mapping

$$h : \langle L, \leq \rangle \to \langle P_\mathbb{Q}, \leq \rangle .$$

We will use the set $P_\mathbb{Q}$ as defined in Lemma 2.18 on p. 29, the countable, dense in itself subset of a perfect set. Note that 0 and $\max V^\infty$ are contained in $P_\mathbb{Q}$ (see Lemma 2.18). We cannot use the function h given in Horn's lemma, but we have to give another one which has the same properties, i.e. that existing infima and suprema are preserved.

The problem with the set $P_\mathbb{Q}$ is that it may be nowhere dense and that Horn's h could produce a value which is not in the set. When adjusting this function we have to take care that infima and suprema are preserved, this is the most difficult part. The first idea of calculating the value of h and if it is not within $P_\mathbb{Q}$, shift it to one side of the gap, does not work, because if the infimum of a sequence is on the left side of a gap, but all the elements are shifted to the right side because the value under h is in the gap, then the infimum will not be preserved. So we have to ensure that the elements of a sequence jump over gaps coming always closer to the limit.

We will solve this problem as follows: We will use the function h to compute an initial value. If it is in a gap we decide where to put it, into the upper or the lower part, depending on the position in the gap: Is the initial value in the lower half of the gap, we put it into the lower part, if it is in the upper half, we put it into the upper part. The lower and upper part are those values of $P_\mathbb{Q}$ which are greater than the upper border (smaller than the lower border) of the gap. Furthermore, we put them always closer to the border than all the previous elements, only dependent on the index in an enumeration of all elements. This way we ensure that finally every gap between list elements and the infimum/supremum will be skipped.

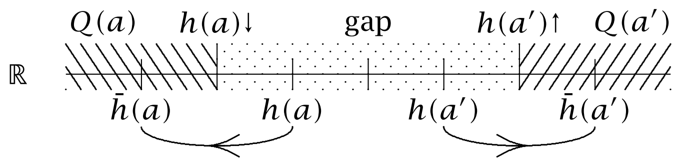

Figure 5.1: From h to $\bar{h}$

For any element $x \in [0,1]$ we define two values: $x\uparrow$ and $x\downarrow$ in the perfect set $P = V^\infty$ as follows: If $x \in P_\mathbb{Q}$ then $x\uparrow$ and $x\downarrow$ are both equal to x. Otherwise $x\uparrow$ is the next larger (in terms of order) element in P and $x\downarrow$ the next smaller one:

$$x\uparrow = \min_{\mathbb{R}}\{p \in P : p > x\}$$
$$x\downarrow = \max_{\mathbb{R}}\{p \in P : p < x\}$$

In the special case of Cantor's middle third set and the representation of its elements in triadic notation we can describe the same operation as follows: Let $x = 0.w1w'$ with $w \in \{0,2\}^*$ and $w' \in \{0,1,2\}^\omega$ (the word w' can be infinitely long), then we define

$$x\uparrow = 0.w0222\ldots$$
$$x\downarrow = 0.w2$$

Again, let

$$L = \{a_0 = \mathbf{0}, a_1 = \mathbf{1}, a_2, \ldots\}$$

and define $\bar{h}(a_0) = 0, \bar{h}(a_1) = \sup V^\infty$. Let

$$h(a_n) = \frac{\bar{h}(a_i) + \bar{h}(a_j)}{2}$$

with a_i the largest member of $\{a_k : k < n\}$ and a_j the smallest member which is $> a_n$ (see Lemma 1.24 on p. 19 for comparison). This is the first proposal for a value of $\bar{h}(a_n)$. If this value is not in $P_\mathbb{Q}$ we have to adjust the value. If $h(a_n) \in P_\mathbb{Q}$ we define $\bar{h}(a_n) = h(a_n)$, otherwise we define a set $Q(a_n)$ of numbers from $P_\mathbb{Q}$ for a_n as follows: If $h(a_n)$ is in the upper half of a gap, i.e. if $|x - x\downarrow| > |x\uparrow - x|$ we have

$$Q(a_n) = \{q \in P_\mathbb{Q} : \forall j < n(a_j > a_n \supset \\ \supset h(a_n)\uparrow < q < \bar{h}(a_j) \wedge q - h(a_n)\uparrow < 1/n)\}$$

and otherwise (lower half)

$$Q(a_n) = \{q \in P_\mathbb{Q} : \forall j < n(a_j < a_n \supset \\ \supset \bar{h}(a_j) < q < h(a_n)\downarrow \wedge h(a_n)\downarrow - q < 1/n)\}.$$

Finally we define $\bar{h}(a_n)$ as one element from $Q(a_n)$.

The following lemma shows that this modified valuation function has the necessary properties.

Lemma 5.4

$$\bar{h}(\inf_L\{|A(t)| : t \in T\}) = \inf_{\mathbb{R}}\{\bar{h}(|A(t)|) : t \in T\}$$

Proof: Let

$$\begin{aligned} a_N &= \inf_L\{|A(t)| : t \in T\} \\ s &= \bar{h}(a_N) \\ q &= \inf_{\mathbb{R}}\{\bar{h}(|A(t)|) : t \in T\} \end{aligned}$$

and suppose that $s \neq q$.

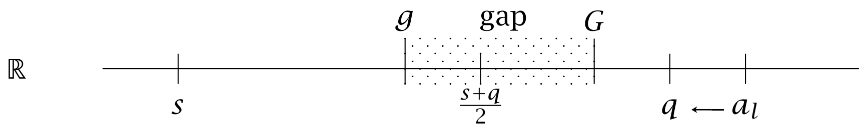

Figure 5.2: The gap between s and q

First we can show as in the proof of Lemma 1.25 that *all* a_i with $a_I > a_N$ are mapped into $(q, 1] \cap P_{\mathbb{Q}}$.

Now construct a sequence of elements a_{l_i} as follows: $a_{l_0} = a_N$, and for all $i > 0$ take the minimal l_i such that

1. $l_i > l_{i-1}$
2. $\forall u < l_i : a_u < a_N \vee a_{l_i} < a_u$

The following properties are easy to verify:

$$\begin{aligned} \lim_{i\to\infty} \bar{h}(a_{l_i}) &= q \\ \lim_{i\to\infty} h(a_{l_i}) &= \frac{q+s}{2} \end{aligned}$$

The former one is obvious since q is the infimum of all the $\bar{h}(a_k)$ (where $a_k > a_N$), the latter one is a consequence from the former, as the values of $\bar{h}$ and h are computed from each other.

From the difference of the limits it follows that there must be a gap (g, G) around $(q+s)/2$, otherwise the $\bar{h}$ values would not be larger than q. I.e., $(q+s)/2 < G \leq q$. Thus, the values $h(a_{l_i})\uparrow$ are less or equal to G.

First assume that $G < q$. We consider the supremum of the sets $Q(a_{l_i})$:

$$\begin{aligned}\sup Q(a_{l_i}) &\leq h(a_{l_i})\uparrow + \frac{1}{l_i} \\ &\leq G + \frac{1}{l_i} \\ &< q \text{ for large enough } i\end{aligned}$$

But, since $\bar{h}(a_{l_i})$ is taken out of the set $Q(a_{l_i})$, it follows that for large enough i, $\bar{h}(a_{l_i}) < q$, which is a contradiction.

So it remains to find a contradiction for the case where $G = q$, i.e. where q is an upper boundary point. Remember, that all the $h(a_{l_i})$ must fall into the upper half of the gap, otherwise the value of $\bar{h}(a_{l_i})$ would be computed from $h(a_{l_i})\downarrow$, immediately generating a contradiction. Thus, the middle of the gap has to be less or equal than $h(a_{l_i})$, and also less or equal than the limes of the $h(a_{l_i})$, which is $(q + s)/2$. So the gap spans the whole interval (s, q), which is a contradiction to the choice of s as an inner point (by construction of $\bar{h}$).

The case of the existential quantifier is treated accordingly.

This concludes the proof that the infima are preserved under $\bar{h}$ and that the valuation given by

$$\mathcal{I}(A) = \bar{h}(|A|)$$

is in fact a correct valuation and a countermodel. □

THEOREM 5.5 (COMPLETENESS OF GÖDEL LOGICS WITH $0 \in V^\infty$)
A formula of Gödel logic is valid in a truth value set whose perfect kernel contains 0 *iff it is derivable from* **H**.

PROOF: We take Takano's proof and instead of using the function h given there we use the function $\bar{h}$ given above for the mapping

$$h : \langle \mathcal{F}/\equiv, \leq \rangle \to \langle P_{\mathbb{Q}}, \leq \rangle \to \langle V^\infty, \leq \rangle \to \langle V, \leq \rangle$$

□

5.4.2 Case $0 \notin V^\infty$, 0 isolated

In the case where 0 is isolated from the perfect kernel we will use the following approach: First the truth value set is transformed into a new one by shifting the minimum of the perfect kernel into 0 and thereby cutting out everything between 0 and the minimum. The resulting truth value set contains 0 in the perfect kernel and thus the completeness result from Theorem 5.5 can be used to obtain a counter model, which will be transformed into a counter model for the original truth value set.

THEOREM 5.6 *A formula is valid in an uncountable truth value set where* 0 *is isolated iff it is derivable from* $\mathbf{H} + \text{ISO}_0$.

PROOF: We will prove that if we have a (possibly) infinite Π and A then either there is a finite subset Π' of Π such that A is provable from Π' in $\mathbf{H} + \text{ISO}_0$ or A is not entailed by Π. This gives strong completeness.

We translate the perfect kernel V^∞ into 0 by subtracting the infimum of V^∞:

$$V_1 = V^\infty - \inf V^\infty$$

(note that V^∞ is the perfect kernel). This shifts the perfect kernel into the origin. Now choose an inner point x of V_1 such that $0 < x < \sup V_1$ and set

$$V' = (V_1 \cap [0, x]) \cup \{1\}$$

This truth value set is a perfect set and 0 is contained in it. Later we will shift the interpretation back into the perfect kernel, but we have to ensure that an unshifted truth value which is less than 1 is not shifted into 1. It could be the case that 1 is contained in the perfect kernel and the unshifted truth value is the supremum of the set V_1, which is less than 1, but when shifted back into the perfect kernel becomes 1. Therefore, it is necessary to cut out a part of the perfect kernel.

Define

$$\Gamma = \{\forall \bar{y}(\neg\forall x A(x, \bar{y}) \supset \exists x \neg A(x, \bar{y})) : A(x, \bar{y}) \text{ formula}\}$$

where $A(x, \bar{y})$ ranges over *all* formulas with free variables x and $\bar{y}$. We consider the entailment relation in V'. Either

$$\Pi, \Gamma \Vdash_{V'} A$$

or

$$\Pi, \Gamma \nVdash_{V'} A.$$

In the former case we know from the strong completeness of $\mathbf{H}$ for $\mathbf{G}$ that there are finite subsets Π' and Γ' of Π and Γ, respectively, such that

$$\Pi', \Gamma' \vdash_{\mathbf{H}} A.$$

Since all the sentences in Γ are provable in $\mathbf{H} + \text{ISO}_0$ we obtain that

$$\Pi' \vdash_{\mathbf{H}+\text{ISO}_0} A.$$

In the latter case there is an interpretation $\mathcal{I}' = \langle D', \mathbf{s}' \rangle$ such that

$$\inf\{\mathcal{I}'(G) : G \in \Pi \cup \Gamma\} > \mathcal{I}'(A).$$

It is obvious from the structure of the formulas in Γ that their truth value will always be either 0 or 1. Combined with the above we know that

$$\mathcal{I}'(G) = 1 \qquad \text{for all } G \in \Gamma.$$

Next we define a function $\phi(x)$ which shifts the interpretation back into the original truth value set V

$$\phi(x) = \begin{cases} 0 & x = 0 \\ x + \inf V^\infty & 0 < x < 1 \\ 1 & x = 1 \end{cases}$$

and define an interpretation $\mathcal{I} = \langle D, \mathbf{s} \rangle$ by

$$\mathbf{s} = \mathbf{s}' \circ \phi, \qquad D = D',$$

i.e. that $\mathbf{s}(P) = \phi(\mathbf{s}'(P))$ for P atomic. Due to the fact that we just shifted the truth value set back and forth, it is obvious that this function in fact maps into V. It remains to show that this interpretation gives a counterexample to $\Pi \Vdash A$, i.e.

$$\inf\{\mathcal{I}(G) : G \in \Pi\} > \mathcal{I}(A).$$

This is a consequence of the following lemma:

LEMMA 5.7

$$\mathcal{I} = \mathcal{I}' \circ \phi$$

PROOF: The proof is done by induction on the complexity of the formula. For $A \wedge B$, $A \vee B$, $A \supset B$ this is obvious. The case of $\exists x A(x)$ is also easy to handle due to the carefully chosen V' such that $\phi(\sup(V' \setminus \{1\})) < 1$. So if $\exists x A(x)$ is evaluated to 1 under $\mathcal{I}'$ then there must be a witness and therefore, it is also evaluated to 1 under $\mathcal{I}$.

The important case is $\forall x A(x)$ where we have to show that by shifting the 0 away (in fact shifting the rest away from 0) we do not change the evaluation. In the case the $\mathcal{I}'(\forall x A(x)) > 0$ it is obvious that

$$\mathcal{I}(\forall x A(x)) = \phi(\mathcal{I}'(\forall x A(x))).$$

In the case where $\mathcal{I}'(\forall x A(x)) = 0$ we observe that since $A(x)$ contains a free variable and therefore,

$$\neg \forall x A(x) \supset \exists x \neg A(x) \in \Gamma$$

we know that

$$\mathcal{I}'(\neg \forall x A(x) \supset \exists x \neg A(x)) = 1$$

and thus that there is a witness c such that $\mathcal{I}'(A(c)) = 0$. Using the induction hypothesis we know that $\mathcal{I}(A(c)) = 0$, too. Thus, we obtain that

$$\mathcal{I}(\forall x A(x)) = 0,$$

which completes the proof. □

Thus we have shown that $\mathcal{I}$ is a counterexample to $\Pi \Vdash_V A$ which completes the proof of the theorem. □

5.4.3 Case $0 \notin V^\infty$, 0 not isolated

We will extend the proof given in Section 5.3 to the case that 0 is not in the perfect kernel V^∞ and it is not isolated. Alternatively, we could say for truth value sets where a neighborhood of 0 exists which is countable and no neighborhood of 0 is singular. We will again establish this result by reducing classical validity of a formula in all finite models to the validity of a formula in this Gödel logic.

Theorem 5.8 *If* $0 \notin V^\infty$ *and* 0 *is not isolated, then* $\mathbf{G}_V$ *is not axiomatizable.*

Proof: In the proof for Theorem 5.3 levels have been defined such that if there are two distinct elements in a level $\leq i$ there is an element in level $i + 1$ which lies strictly between those two elements in the ordering given by $\prec$. In the countable case we could carry out this construction directly, in the present case we have to ensure that this construction is carried out in a countable surrounding of 0. This is done by adding a sequence decreasing to 0 by using

$$\forall u \neg\neg P(u) \wedge \neg \forall u P(u)$$

which expresses that all the $P(u)$ are greater than 0, but the infimum is 0. Then for all u the construction from Theorem 5.3 is carried out in the interval $[0, u]$ (interpreted as the interval $[0, \mathcal{I}(P(u))]$) *in parallel*, and if it terminates in one of these intervals, the next level is empty in *all* intervals/parallel constructions.

We define A^g as follows: Let P be a unary and L be a ternary predicate symbol not occurring in A and let $Q_1, \dots, Q_n$ be all the predicate symbols in A. We use the following abbreviations:

$$x \in_u y \equiv \neg\neg L(x, y, u)$$
$$x \prec y \equiv P(x) \prec P(y) \equiv (P(y) \supset P(x)) \supset P(y)$$

Note that for any interpretation $\mathcal{I}$, $\mathcal{I}(x \in_u y)$ is either 0 or 1, and as long as $\mathcal{I}(P(x)) < 1$ (in particular, if $\mathcal{I}(\exists z P(z)) < 1$), we have $\mathcal{I}(x \prec y) = 1$ iff $\mathcal{I}(P(x)) < \mathcal{I}(P(y))$. Let

$$A^g \equiv S \wedge \forall u \neg\neg P(u) \wedge \neg \forall u P(u) \wedge \wedge \forall u \Big\{ c \in_u 0 \wedge u \in_u 0 \wedge \neg P(c) \wedge \\ \wedge \forall i [\forall x, y \forall j, k \exists z \, Levels \vee \forall u' \forall x \neg(x \in_{u'} s(i))] \Big\} \supset \\ \supset (A' \vee \exists u \, P(u))$$

where S is the conjunction of the standard axioms for 0, successor s and $\leq$, with double negations in front of atomic formulas,

$$Levels \equiv \begin{array}{c} j \leq i \wedge x \in_u j \wedge k \leq i \wedge y \in_u k \wedge x \prec y \supset \\ \supset (z \in_u s(i) \wedge x \prec z \wedge z \prec y) \end{array}$$

and A' is A where every atomic formula is replaced by its double negation, and all quantifiers are relativized to the predicate

$$R(i) \equiv \exists u \exists x (x \in_u i).$$

As in the mentioned proof, intuitively L is a predicate that divides a subset $[0, u]$ of the domain into levels, and $x \in_u i$ means that x is an element of level i in the interval $[0, u]$. If the construction of the middle element cannot be done at one level in one neighborhood, the next level in *all* neighborhoods is empty, this is accomplished by the $\forall u' \forall x \neg(x \in_{u'} s(i))$. Again, the required points to fill a level can only be found for finitely many levels in parallel if no neighborhood of 0 is uncountable. By relativizing the quantifiers in A to the indices of non-empty levels, we in effect relativize to a finite subset of the domain. We make this more precise:

Suppose A is classically false in some finite structure $\mathcal{I}$. Without loss of generality, we may assume that the domain of this structure is the naturals $0, \ldots, n$. We extend $\mathcal{I}$ to a $\mathbf{G}_\mathbb{R}$ interpretation $\mathcal{I}^g$ with domain $\mathbb{N}$ as follows: Since V contains infinitely many values, we can choose c, L and P so that $\forall u \exists x (x \in_u i)$ is true for $i = 0, \ldots, n$ and false otherwise, and so that $\sup \mathrm{Distr}_{\mathcal{I}^g} P(x) < 1$. The number theoretic symbols receive their natural interpretation. The antecedent of A^g clearly receives the value 1, and the consequent receives $\sup \mathrm{Distr}_{\mathcal{I}^g} P(x) < 1$, so $\mathcal{I}^g \not\models A^g$.

Now suppose that $\mathcal{I} \not\models A^g$. Then $\mathcal{I}(\exists x \, P(x)) < 1$ and $\sup \mathrm{Distr}_\mathcal{I} P(x) < 1$. In this case, $\mathcal{I}(x \prec y) = 1$ iff $\mathcal{I}(P(x)) < \mathcal{I}(P(y))$, so $\prec$ defines a strict order on the domain of $\mathcal{I}$. It is easily seen that in order for the value of the antecedent of A^g under $\mathcal{I}$ to be greater than that of the consequent, it must be $= 1$ (the values of all subformulas are either $\leq \sup \mathrm{Distr}_\mathcal{I} P(x)$ or $= 1$). For this to happen, of course, what the antecedent is intended to express must actually be true in $\mathcal{I}$, i.e. that $x \in_u i$ defines a series of disjoint levels and that for any u and for any i, either level $i+1$ is empty or for all x, y s.t. $x \in_u j$, $y \in_u k$ with $j, k \leq i$ and $x \prec y$ there is a z with $x \prec z \prec y$ and $z \in_u i+1$. To see this, consider the relevant part of the antecedent,

$$B = \forall i [\forall x, y \forall j, k \exists z \, Levels \vee \forall u' \forall x \neg (x \in_{u'} s(i))].$$

If $\mathcal{I}(B) = 1$, then for all i, either

$$\mathcal{I}(\forall x, y \forall j, k \exists z \, Levels) = 1$$

or $\mathcal{I}(\forall u' \forall x \neg (x \in_{u'} s(i))) = 1$. In the first case, we have

$$\mathcal{I}(\forall z \, Levels) = 1$$

for all x, y, j, and k. Now suppose it were not the case that for some z, the $\mathcal{I}(Levels) = 1$, yet $\mathcal{I}(\exists z \, Levels) = 1$. Then for at least some z the value of that formula would have to be $> \sup \mathrm{Distr}_{\mathcal{I}} P(z)$, which is impossible. Thus, for every x, y, j, k, there is a z such that $\mathcal{I}(Levels) = 1$. This means that for all x, y s.t. $x \in_u j$, $y \in_u k$ with $j, k \leq i$ and $x \prec y$ there is a z with $x \prec z \prec y$ and $z \in_u i+1$.

In the second case, where $\mathcal{I}(\forall u' \forall x \neg (x \in_{u'} s(i))) = 1$, we have that $\mathcal{I}(\neg (x \in_{u'} s(i))) = 1$ for all x and all u', hence for all u' $\mathcal{I}(x \in_{u'} s(i)) = 0$ and level $s(i)$ is empty in all neighborhoods $[0, u]$ of 0.

Since there is a neighborhood of 0 which does not contain a dense subset, from some finite level $s(i)$ onward the levels must be empty. Of course, $i > 0$ since $c \in 0$. Thus, A is false in the classical interpretation $\mathcal{I}^c$ obtained from $\mathcal{I}$ by restricting $\mathcal{I}$ to the domain $\{0, \ldots, i\}$ and $\mathcal{I}^c(Q) = \mathcal{I}(\neg\neg Q)$ for atomic Q. □

5.4.4 First-order Gödel logics with Δ

If the language lets us express the property of being 1, i.e. if the language contains Δ, we can express an infinite ascending sequence of truth values with supremum equal to 1:

$$\forall x \neg \Delta A(x) \wedge \Delta \exists x A(x)$$

This formula expresses that for all x the truth value of $A(x)$ is less than 1, but the truth value of $\exists x A(x)$ is 1.

Similar to the case above we cannot put the countable dense linear subset arbitrarily in the perfect set, but the completion of the subordering must have 0 as smallest and additionally 1 as largest element. This means for the perfect set that 0 and 1 must be contained in the perfect set. If these conditions are fulfilled we can again use the adapted function $\bar{h}$ to show the completeness.

Theorem 5.9 *A formula of first-order language with Δ is valid in a truth value set whose perfect kernel contains* 0 *and* 1 *iff it is provable in* $\mathbf{H}\Delta$.

Proof: Analogous to the proof of Theorem 5.5. □

The extension to the mixture of cases where either 0 or 1 or both are isolated or one of them is in the perfect kernel and the other one is isolated is straightforward using the proofs for the case without Δ, so we obtain

Theorem 5.10 *The following axiom systems are complete for the respective logics:*

$$\begin{array}{rl} 0 \in V^\infty,\ 1 \textit{ isolated} & \mathbf{H}\Delta + \textsc{iso}_1 \\ \textit{0 isolated},\ 1 \in V^\infty & \mathbf{H}\Delta + \textsc{iso}_0 \\ V^\infty \neq \emptyset,\ 0, 1 \textit{ isolated} & \mathbf{H}\Delta + \textsc{iso}_0 + \textsc{iso}_1 \end{array}$$

Proof: Analogous to the proof of Theorem 5.6. □

CHAPTER 6

First-order entailment

If we proceed to full first-order we can use Takano's proof and its extension given in the previous chapter, as they provide a strong completeness result, and therefore, all complete, i.e. complete recursive axiomatizable Gödel logics are also compact.

We will now show that the entailment relation of a Gödel logic, which is not r.a., is not compact.

LEMMA 6.1 *If a Gödel logic permits the definition of finiteness, its entailment relation cannot be compact.*

PROOF: We will prove this lemma by giving formulas which express that the domain has at least n elements for every n. All these formulas entail a formula expressing the fact that the domain is infinite. No finite subset of these formulas entails it, thus the entailment relation is not compact. More precisely, let $equ(R)$ be the axioms asserting that the binary predicate symbol R is an equivalence relation:

$$\begin{aligned} equ(R) = & \forall x \neg\neg R(x,x) \wedge \\ & \wedge \forall x \forall y (\neg\neg R(x,y) \supset \neg\neg R(y,x)) \wedge \\ & \wedge \forall x \forall y \forall z (\neg\neg R(x,y) \wedge \neg\neg R(y,z) \supset \neg\neg R(x,z)) \end{aligned}$$

and let

$$F_n = equ(R) \supset \exists x_1 \ldots \exists x_n \bigwedge_{i \neq j} \neg R(x_i, x_j)$$

then F_n expresses the fact that there are at least n elements in the domain. Furthermore let

$$\begin{aligned}A = S \wedge \forall u \neg\neg P(u) \wedge \neg \forall u P(u) \wedge \\ \wedge \forall u \Big\{ c \in_u 0 \wedge u \in_u 0 \wedge \neg P(c) \wedge \\ \wedge \forall i \left[\forall x, y \forall j, k \exists z Levels \vee \forall u' \forall x \neg(x \in_{u'} i) \right] \Big\}\end{aligned}$$

be similar to the formula we know from the proof of Theorem 5.8, with *Levels* as on p. 52 defined. Now consider the following entailment:

$$F_1, F_2, \ldots \Vdash A \supset \exists u \exists i \forall x \neg(x \in_u i)$$

As we have explained in the proof of Theorem 5.3 and Theorem 5.8 in any countable truth value set the above relation will hold for any interpretation. Now assume that there is a finite subset F' of $\{F_1, F_2, \ldots\}$ such that

$$F' \Vdash A \supset \exists u \exists i \forall x \neg(x \in_u i).$$

But as the set F' only asserts the existence of a finite number N of objects we can define an interpretation $\mathcal{I}$ such that the process described in *Levels* will only collapse after N steps and thus the set F' does *not* entail the given formula.

This proves that the entailment relation for Gödel logics based on truth value sets where there is a neighborhood of 0 which is countable and 0 is not isolated, is not compact. This includes the case where the truth value set is countable and 0 not isolated. The case were 0 is isolated in a countable truth value set is treated similar, with A from above replace by the more simple formula A from Theorem 5.3. □

Theorem 6.2 *The entailment relation of a Gödel logic is compact if and only if the logic admits a complete recursive axiomatization.*

Proof: We have shown that in all logics which do not admit a complete recursive axiomatization, finiteness can be defined, and using the above lemma we conclude that their entailment relation is not compact. □

CHAPTER 7

Conclusion

In the last years the area of many-valued logics, and in particular Gödel logics, have attracted many scientists and substantial improvements in the understanding of this large family have been achieved. In Gödel logics, the entailment relations in the propositional case have been fully characterized [BZ98]. The most important quantified propositional Gödel logics have been analyzed. The proof-theoretic foundations of some important first-order and propositionally quantified Gödel logics have been established [Avr91, BCF].

From every solved problem many new emerge, thus, there are a host of interesting problems which still await settling. We would like to mention a few which we consider as of primary interest for the understanding of Gödel logics:

Topological characterization Gödel logics are defined extensionally as the set of valid sentences over a certain truth value set. As we have shown, many of these logics coincide. The development of standard models for Gödel logics overcoming this peculiarity will provide enormous insight.

Relation to Kripke semantics As intermediate logics, i.e. intermediate between intuitionistic and classical logic, the relationship to Kripke models, which form the standard models for intuitionistic logic, is not well understood. Only in very specific cases (see Section 1.3, and [Cor89, Cor92]) there is a known relationship.

Number of Gödel logics The number of different Gödel logics is still unknown in the first-order case. We only know that there are at least countably many, but there may exist uncountably many. This question has only been settled for the entailment relation and quantified propositional Gödel logic.

Expressiveness Strongly connected to the above question is the question for the limits of expressiveness of Gödel logics. This includes the comparison of expressiveness of formulas and rules, which are more expressive. E.g., the strong soundness of the Takeuti-Titani rule expresses that the truth value set is everywhere dense, but there is no first-order formula with this property.

Monadic fragments From a computational point of view the monadic fragment often is one of the most interesting fragments of a logic, as this fragment naturally turns up in logic programming and formalizations. A different view of the monadic fragment allows the interpretation of the predicates as fuzzy sets, allowing to work with this well accepted notion.

All these questions, and many more, will be considered and the respective answers will provide fundamental insights into many-valued logics and Gödel logics.

Finally, the table on the following page presents an overview of the obtained results.

Survey of results

Propositional Logic

V infinite	$\mathbf{LC} = \mathbf{H}^0, \mathbf{H}\Delta^0$ complete for the logic Theorem 3.4, p. 32, Theorem 3.10, p. 35
V finite (n)	$\mathbf{LC}_n = \mathbf{H}^0_n, \mathbf{H}\Delta^0_n$ complete for the logic Theorem 3.9, p. 34, Theorem 3.11, p. 35

Propositional Entailment

V finite	$\mathbf{G}^0_V$ compact	Theorem 4.5, p. 38
V countable	$\mathbf{G}^0_V$ not compact	Theorem 4.7, p. 40
V uncountable	$\mathbf{G}^0_V$ compact	Theorem 4.6, p. 39

First-order logic

V finite (n)	$\mathbf{H}_n$ complete for the logic Theorem 5.1, p. 42
V countable	not recursively enumerable Theorem 5.3, p. 43
$V^\infty \neq \emptyset$, $0 \in V^\infty$	$\mathbf{H}$ complete for the logic Theorem 5.5, p. 49
$V^\infty \neq \emptyset$, 0 isolated	$\mathbf{H} + \textsc{iso}_0$ complete for the logic Theorem 5.6, p. 50
$V^\infty \neq \emptyset$, $0 \notin V^\infty$, 0 not isolated	not recursively enumerable Theorem 5.8, p. 52

First-order logic with Δ

V finite (n)	$\mathbf{H}\Delta_n$ complete for the logic Theorem 5.2, p. 43
$0, 1 \in V^\infty$	$\mathbf{H}\Delta$ complete for the logic Theorem 5.9, p. 55
$0 \in V^\infty$, 1 isolated	$\mathbf{H}\Delta + \textsc{iso}_1$ complete for the logic Theorem 5.10, p. 55
0 isolated, $1 \in V^\infty$	$\mathbf{H}\Delta + \textsc{iso}_0$ complete for the logic Theorem 5.10, p. 55
$V^\infty \neq \emptyset$, 0, 1 isolated	$\mathbf{H}\Delta + \textsc{iso}_0 + \textsc{iso}_1$ complete Theorem 5.10, p. 55
$V^\infty \neq \emptyset$, $1 \notin V^\infty$, 1 not isolated	not recursively enumerable Theorem 5.8, p. 52

First-order entailment

recursive axiomatizability = compactness

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Curriculum Vitae

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Education

- April 2003: Application for PhD at Vienna University of Technology.
- since 1996: PhD studies at Vienna University of Technology.
- 27. June 1996: Master degree in mathematics with excellence.
- since 1990: Study of classical philology at the University of Vienna.
- since 1990: Study of physics at Vienna University of Technology.
- April-Sept. 1991: Working in Austrian Survey Office.
- 1989–1996: Study of mathematics at Vienna University of Technology.
- 1989: Participation at the Austrian Mathematic Olympics.
- 1977–1989: Basic education in Vienna.

Additional qualification

- since 1998: Training as “Initiateur et Moniteur d’Alpinisme” in France at the École Nationale de Ski et d’Alpinisme
- 1996-1997: Training as federal mountaineering instructor (staatliche Hochalpinlehrwart)
- 1994-1995: Training as federal ski touring instructor (staatlicher Skitourenlehrwart)
- 1990-1992: Training as Youth Guide of the Austrian Alpine Club

Participation in scientific projects

Participation in the following research projects of the Austrian Research Fund (FWF):

- P11934-MAT Cut Elimination and Cut Introduction
- P14126-MAT Tools for the automated analysis of proofs
- P15477-MAT Gödel Logics: Propositional Quantifiers and Concurrency
- P16254-N05 Proof Transformation by Resolution

Related activities

- Organization of ESSLLI 2003 in Vienna
- Organization of CSL 2003 in Vienna
- Organization of Logic Colloquium 2001 in Vienna
- Organization of KGS’s participation at ScienceWeek 2001
- Organization of KGS’s participation at ScienceWeek 2000
- since 2000 Secretary of the Kurt Gödel Society
- 1999–2000 Vice secretary of the Kurt Gödel Society
- Head of the Viennese Youth Leader Training Program of Austrian Alpine Club.

Publications and Lectures

- N. Preining. Sketch-as-proof, a proof-theoretic analysis of axiomatic projective geometry. Master’s thesis, University of Technology, Vienna, Austria, 1996.
- N. Preining. Sketch-as-proof. In G. Gottlob, A. Leitsch, and D. Mundici, editors, *Computational Logic and Proof Theory, Proc. 5^{th} Kurt Gödel Colloquium* KGC’97, Lecture Notes in Computer Science 1289, pages 264–277, Vienna, Austria, 1997. Springer.

- N. Preining. How fast are sketches as proofs. Talk at the LC'98, August 1998.
- N. Preining. An application of cut elimination in projective geometry. Talk at the RISC, March 1999. Hagenberg, Austria.
- N. Preining. Using Tibetan language with $\TeX/\Omega$. Invited talk at the 4th International Symposium on Multilingual Information Processing, March 2000. Tsukuba, Japan.
- N. Preining. Herbrand disjunctions and sketches. Talk at the First Moscow-Vienna Logic Meeting, December 2000. Steklov Institute, Moscow, Russia.
- N. Preining. Sketches in affine geometry. Talk at the CALCULEMUS'01, IJCAR 2001, June 2001. Siena, Italy.
- M. Baaz, A. Ciabattoni, N. Preining, and H. Veith. A guide to quantified propositional Gödel logic. IJCAR workshop QBF 2001, June 2001. Siena, Italy.
- N. Preining. What to do with Herbrand disjunctions in geometry. Talk at LOGSOFCOM'01, November 2001. Gargnano, Italy.
- N. Preining. Characterization of decidability of Gödel logics. Talk at the Second Moscow-Vienna Logic Meeting, April 2002. Steklov Institute, Moscow, Russia.
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